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PHASE-PLANE METHODS

Yun-Lan Hsiung

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PHASE-PLANE METHODS

by

Yun-Lan Hsiung

October 1969

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Phase-Plane Methods

by

Yun-Lan Hsiung
Lieutenant Commander, Chinese Navy
B. S., Chinese Naval Academy, 1954

Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

The phase-plane method is a graphical method for linear and non-linear second-order systems. There are many techniques for obtaining the phase portrait which consists of a number of phase trajectories on the phase-plane. From the summary and discussion the isocline method is a most general and useful method. The only difficulty is in the labor required. Solution by digital computer reduces this difficulty and makes the isocline method a much more useful method in non-linear systems analysis and design applications.

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I. INTRODUCTION

A. CONDITIONS FOR LINEARITY AND DEFINITION OF NON-LINEARITY

It is useful to define linearity and then any system, component, or phenomenon, which does not satisfy this definition will be considered non-linear. The simple meaning of linearity is that if the relationship between output y and input x is plotted the result is a straight line. Or that if the output is proportional to the input the process must be linear. So we can simply say the essential condition for linearity is that the law of superposition applies, and linearity is a specification which therefore limits the field of activity. Non-linearity is a "non-specification" and the field is unbounded, linearity being a particular part of it.

B. CLASSES OF NON-LINEARITY

Non-linearity in physical systems cannot be avoided, and in fact can sometimes be used to improve the system's performance. The non-linearities encountered in control systems may be classified as accidental non-linearities and intentional non-linearities. The accidental class consists of those non-linearities which are inherent in the components used in the system. Some of these are coulomb friction, stiction, backlash, binding, dead zone, saturation, etc. Their existence is accidental in the sense that they are not deliberately inserted in the system. Intentional non-linearities are those which are introduced deliberately, either to obtain desired performance features or for weight, space, or economic advantages.

Clearly the simplest class of non-linearity is the instantaneous, a member of which only changes its input u(t) to some output x(t) without any delay, the ratio x(t)/u(t) being a function of u(t). Thus x(t) = f[u(t)] and it can be represented in schematic form (see Fig.(1-1)), having a transfer function f(z)/z which is independent of time. Saturating amplifiers and relays are examples of instantaneous non-linearities. In practice, of course, such a non-linearity will almost always be associated with some linear (or non-linear) weighting function resulting from inertia and damping as in Fig.(1-2), so that

$$x(t) = f[u(t)] \tag{1-1}$$

In general, the instantaneous non-linear characteristic is combined with inertia and damping in the same component, in which case it cannot be separated as in Fig.(1-2).

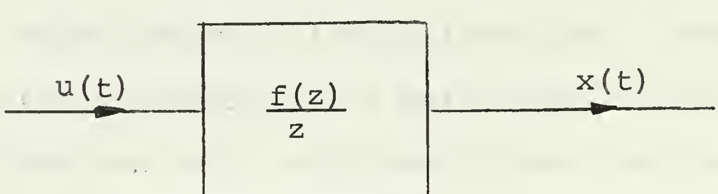


Fig. 1-1. A transfer function $\frac{f(z)}{z}$ which is independent of time.

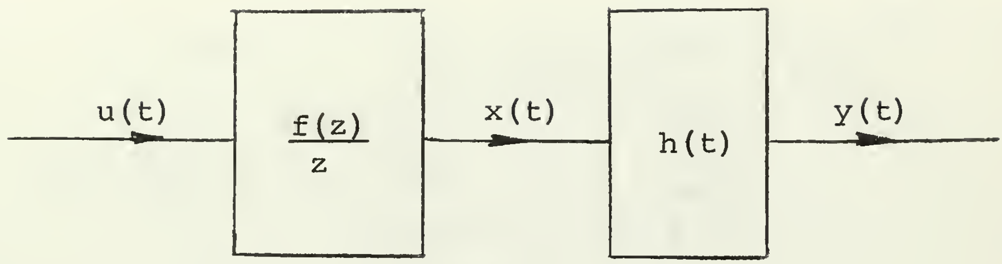


Fig. 1-2. The instantaneous non-linear characteristic is combined with inertia and damping cannot be separated as in figure.

By convolution integral

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau \text{ , and}$$

changing the variable gives

$$y(t) = \int_{0}^{t} x(t-\tau_1)h(\tau_1)d\tau_1 .$$

Assume

$$x < 0 \quad \text{for} \quad t < 0$$

so that

$$x(t-\tau_1) = 0 \quad \text{for} \quad \tau_1 \geq t.$$

Then

$$y(t) = \int_0^\infty x(t-\tau_1)h(\tau_1)d\tau_1$$

$$= \int_0^\infty f[u(t-\tau_1)]h(\tau_1)d\tau_1$$

$$= \int_0^t K[u(t-\tau_1),\tau_1]d\tau_1 \tag{1-2}$$

$K[u(t-\tau_1),\tau_1]$ is called the <u>Kernel function</u> and is seen to depend on both the dynamics of the system, $f(z)/z$ and $h(t)$, and the characteristics of the input process, $u(t)$.

<u>Zadeh</u> has classified non-linear systems on the basis of multiway integrals and Kernel functions. Thus those that can be represented by a single-way integral belong to the first class, η_1, which therefore includes the subclass of all linear systems and also the degenerate zero-order class of instantaneous non-linearities.

Then the first-order non-linear differential equation is

$$\frac{dx}{dt} + F_o(x) = F_1[u(t)]$$

or

$$\frac{dx}{dt} = F_1[u(t)] - F_o(x).$$

Integrating gives

$$x(t) = \int_0^\infty K[u(t-\tau),\tau]d\tau$$

Hence, the second-order non-linear differential equation is

$$f_2\left(\frac{d^2x}{dt^2}\right) + f_1\left(\frac{dx}{dt}\right) + f_o(x) = u(t)$$

$$f(\ddot{x}, \dot{x}, x) = u(t).$$

Integrating twice gives

$$x(t) = \int_o^\infty \int_o^\infty K[u(t-\tau_1),\ u(t-\tau_2),\tau_1,\tau_2]d\tau_1 d\tau_2.$$

So, the second class, η_2, contains all those for which the input/output relationship can be expressed as a 2-way integral and this includes the degenerate cases belonging to class η_1.

The nth class, η_n, contains all those systems for which the input/output relationship is a n-way integral, which means that they can be represented by nth order non-linear differential equations.

Zadeh's classification is an important property in considering random processes in the presence of noise.

C. NON-LINEAR ANALYSIS

Many methods used in the analysis of non-linear problems are effective only for limited types of problems. A few of them have been developed which have more general applicability to non-linear control systems:

1. Linear approximations.

2. Graphical method - The phase-plane.

3. Piecewise linear approximation.

4. Numerical methods.

5. Describing function.

6. The analog/digital computer.

D. PHASE-PLANE ANALYSIS OF NONLINEAR SYSTEMS

The phase-plane method is a graphical method. There are many techniques for obtaining the phase portrait which consists of a number of phase trajectories on the $\dot{x}$ vs x plane or in some plane related to this plane through a convenient transformation of variables. After the phase portrait is obtained, by means of it, analysis of the following aspects and items is possible:

1. Computation of the exact transient response for a given set of initial conditions.

2. Prediction of the type of transient response (oscillatory or monotonic) with a certain range of initial conditions.

3. Estimates of the amount of overshoot in response to a step input of given magnitude.

4. Prediction of the system stability.

5. Prediction of the existence and/or amplitude of limit cycles.

6. Prediction of the existence of various modes of operation.

7. Study of the effect of a given type of non-linearity on the performance of the system as compared with a linear system, with a system having a different type of non-linearity, or with a system which is non-linear in the same sense but in different degree.

8. Attempts to answer questions such as:

a. Does this non-linearity improve performance?

b. What changes can be made to suppress the limit cycle?

c. How can we adjust this system to make it meet specifications?

As stated above, we can understand a graphical method; it is usually relatively simple to utilize, and may be especially useful as an exploratory tool when a non-linear equation is first being attacked. There are many ways to obtain phase trajectories on the phase-plane. In the next chapter, the most useful methods will be summarized and compared. The advantages and disadvantages of the various methods will be pointed out.

II. SUMMARY OF USEFUL GRAPHICAL METHODS FOR CONSTRUCTING PHASE TRAJECTORIES ON THE PHASE-PLANE

A. ISOCLINE METHOD

1. Method-description

The free motion of any second-order non-linear system can be described by an equation of the form:

$$\ddot{x} + \Phi(x,\dot{x})\dot{x} + \Psi(x,\dot{x})x = 0$$

where

$$\dot{x} = \frac{dx}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2}$$

Let

$$y = \frac{dx}{dt} \tag{2-1}$$

then

$$\ddot{x} = \frac{dy}{dt} = - \Phi(x,y)y - \Psi(x,y)x \tag{2-2}$$

Equation (2-2) divided by (2-1) yields

$$\frac{dy}{dx} = - \frac{\Phi(x,y)y + \Psi(x,y)x}{y} .$$

Let

$$\frac{dy}{dx} = M \quad \text{where } M \text{ is a constant.}$$

Then

$$My = \Phi(x,y)y + \Psi(x,y)x \tag{2-3}$$

Equation (2-3) is the equation of isoclines.

NOTES: a. If $\Phi(x,y)$ and $\Psi(x,y)$ are constants, then isoclines are straight lines. Otherwise, isoclines will be curves.

b. For a first-order equation:

$$\frac{dx}{dt} = f(x,t)$$

Isoclines will be $f(x,t) = M$.

c. Equations more than second-order need some transformation technique, so this method is good only for second-order systems.

d. After obtaining the isoclines, phase trajectories are plotted on the phase-plane as follows:

M_1, M_2, M_3 --- represent the slope of each isocline. Start from same initial condition point, as in Fig. 2-1 point A. From point A draw dotted lines parallel to M_1, M_2. These dotted lines intersect the M_2 isocline at a and be. B is middle point of ab. Continuing in this way, obtain C, D ... points. Connect A, B ... to obtain a phase trajectory from the given initial condition A on the phase-plane.

2. <u>Example</u>

Consider a viscous-damped servo

$$J\frac{d^2\theta_o}{dt} + F\frac{d\theta_o}{dt} + K\theta_o = 0.$$

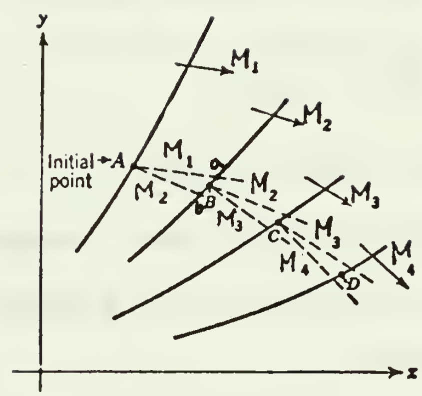


Fig. 2-1. Construction of a trajectory using isoclines.

Dividing through by J:

$$\frac{d^2\theta_o}{dt^2} + \frac{F}{J}\frac{d\theta_o}{dt} + \frac{K}{J}\theta_o = 0. \tag{2-4}$$

Substituting

$$\omega_n = \sqrt{\frac{K}{J}}$$

and

$$c = \frac{F}{2\sqrt{KJ}}$$

into equation (2-4):

$$\frac{d^2\theta_o}{dt^2} + 2c\omega_n \frac{d\theta_o}{dt} + \omega_n^{\,2}\theta_o = 0.$$

Let

$$\frac{d\theta_o}{dt} = z$$

then

$$\frac{dz}{d\theta_o} = \frac{d}{d\theta_o}\left(\frac{d\theta_o}{dt}\right) = \frac{1}{z}\ \frac{d^2\theta_o}{dt^2}$$

so

$$z = \frac{dz}{d\theta_o} + 2c\omega_n z + \omega_n^{\,2}\theta_o = 0.$$

Let

$$M = \frac{dz}{d\theta_o}$$

Isocline equations will be

$$ZM = -2c\omega_n z - \omega_n^2 \theta_o. \tag{2-5}$$

If $c = 0.25$ (linear servo system)

and $\omega_n = 1$,

equation (2-5) will be $ZM = -0.5z - \theta_o$

If M is equal to 2,1,0, -0.5, -1, -2 we obtain straight radial lines (isoclines) as in Fig. 2-2.

Assume an initial condition represented by A in the figure, namely an output position $\theta_o = -2.7$ and a position rate of change $\frac{d\theta_o}{dt} = 0$, we can plot the phase trajectory as in Fig. 2-2.

3. <u>Discussion</u>

a. From Fig. 2-2 it is seen that by plotting some isoclines on the phase-plane (more or less depending on desired accuracy) then any phase trajectory is easily plotted from given initial conditions on the phase-plane.

b. By inspection of a few trajectories the system's behavior is determined. For example: In Fig. 2-2 this system has damped oscillations and is a stable system. Point B is an oscillation max. ($\theta_o = 1.2$).

c. In this example, as applied to the case of a linear servo system (C is constant), isoclines are straight lines, so it is simple. If C is not constant more labor is required.

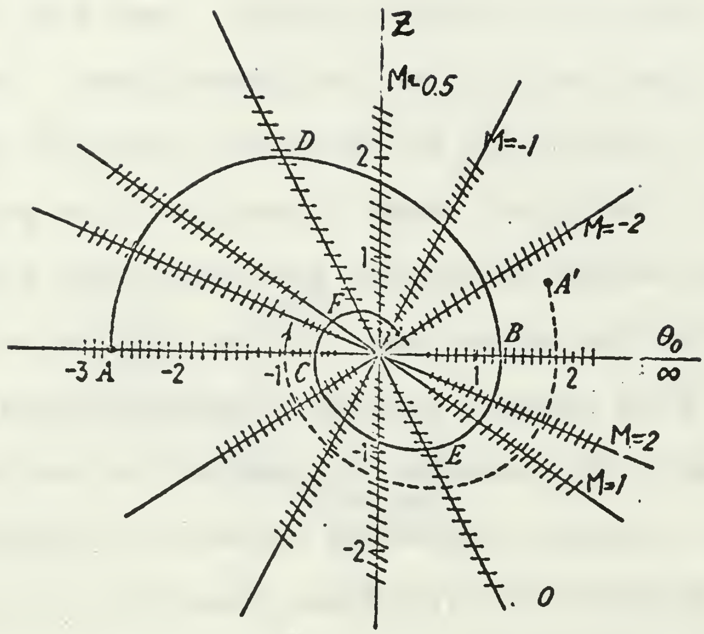


Fig. 2-2. Phase-plane plot of damped oscillation obtained by iso- cline method for a servomechanism.

B. LIÉNARD METHOD

1. Method Description

Liénard's method is a means for drawing a direction field in the phase-plane to guide the tracing of the trajectories of a differential equation.

Using the isocline method, (see Fig. 2-3), draw the zero-slope isocline on the phase-plane. Let P be any point of the plane at which the slope of the direction field is desired. From P draw the line parallel to the z-axis, which intersects the zero-slope isocline at Q. Let R be the projection of Q on the z-axis. Now, draw a short line segment through P, perpendicular to the straight line RP. This segment is tangent to the direction field of the integral curves of certain differential equations.

In practice, graphical construction of the entire direction field is quite easy since the short segments at all the points on the extension of the line PQ can be drawn by using a compass centered at the one point R. Note that this construction does, indeed, result in $\alpha = 0$ everywhere on the zero-slope isocline.

In Fig. 2-3 the two angles α and β are equal because of the perpendicularity of their sides at P and Q. The tangent of each angle can be evaluated in terms of the coordinates of the points P, Q and R and the ratio of lengths of unit distance along the two axes. This length ratio is defined by

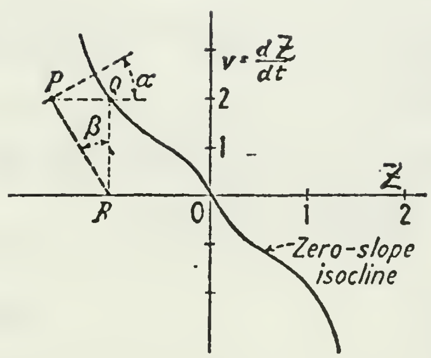


Fig. 2-3. Liénard's construction of phase-plane direction field.

$$K = \frac{\text{unit distance for } v}{\text{unit distance for } z} \tag{2-6}$$

$$\tan \alpha = K \frac{dv}{dz} = KM \tag{2-7}$$

where M is the slope of the integral curves and/or of their direction field

$$\tan \beta = \frac{QP}{QR} = \frac{1}{K}\left(\frac{Z_Q - Z_P}{V_Q - V_P}\right) = \frac{f(V_Q) - Z_P}{KV_Q} \tag{2-8}$$

The expression $Z_P = f(V_Q)$ is the relation defining the zero-slope isocline.

Since $V_Q = V_P$, P is any point of the phase-plane. The subscript can be dropped, giving

$$\tan \beta = \frac{f(V) - Z}{KV} \tag{2-9}$$

since

$$\tan \alpha = \tan \beta$$

then

$$\tan \alpha = KM = \tan \beta = \frac{f(V) - Z}{KV}$$

$$M = \frac{f(V) - Z}{K^2 V} \tag{2-10}$$

$$M = \frac{dv}{dz}$$

$$M \frac{dz}{dt} = \frac{dv}{dz} \ \frac{dz}{dt} = \frac{d^2 z}{dt^2} \tag{2-11}$$

Combining equation (2-10) and (2-11)

$$K^2 \frac{d^2z}{dt^2} - f\left(\frac{dz}{dt}\right) + z = 0 \qquad (2\text{-}12)$$

Any servo which obeys a differential equation of this form has a phase-plane representation with a direction field which can be constructed by Liénard's method.

Note that the coefficient of z is unity and that K specifies the ratio of vertical-to-horizontal unit distances on the phase-plane. The zero-slope isocline is found directly from the differential equation by setting $\frac{d^2z}{dt^2}$ equal to zero, since the isocline is the line for which M = 0. So equation (2-11) gives $\frac{d^2z}{dt^2} = 0$.

2. Example

Using the same example as in A, a linear servomechanism having a constant damping coefficient C = 0.25, and natural frequency ω_n = 10 radian/sec.

$$\frac{d^2\theta_o}{dt^2} + 2c\omega_n \frac{d\theta_o}{dt} + \omega_n^2\theta_o = 0$$

Changing to the required form (compared to Eq.(2-12))

$$\frac{1}{\omega_n^2} \frac{d^2\theta_o}{dt^2} + \frac{2c}{\omega_n} \frac{d\theta_o}{dt} + \theta_o = 0$$

$$\frac{1}{100} \frac{d^2\theta_o}{dt^2} + \frac{1}{20} \frac{d\theta_o}{dt} + \theta_o = 0$$

$$K = \frac{1}{\omega_n} = 0.1$$

$$f\left(\frac{d\theta_o}{dt}\right) = -\frac{2c}{\omega_n}\frac{d\theta_o}{dt} = -\frac{1}{20}\frac{d\theta_o}{dt} = -\frac{V}{20}$$

from which the equation for the zero-slope isocline is obtained.

$$\theta_o = -\frac{V}{20}$$

(See Figs. (2-4) and (2-5)).

3. <u>Discussion</u>

a. Any system which has a differential equation like Eq. (2-12) can use Liénard's method to construct a phase-plane direction field. After the direction field is constructed, one can draw any phase trajectory from a given initial condition on the phase-plane.

b. As compared with the isocline method in Section A. If the differential equation can be manipulated to (2-12) form, a lot of labor may be saved, and one gets the same result. It seems better than the isocline method in this particular problem.

c. If C is not a constant, then the zero-slope isocline will not be a straight line.

d. Liénard's method (even for drawing an individual trajectory) is not applicable to all second-order non-linear differential equations.

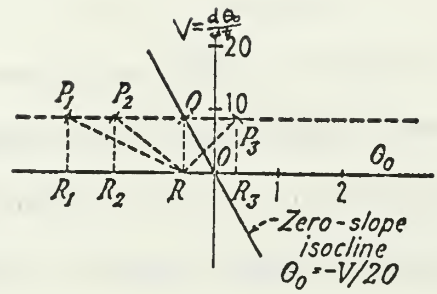


Fig. 2-4. Liénard's construction of the phase-plane direction field.

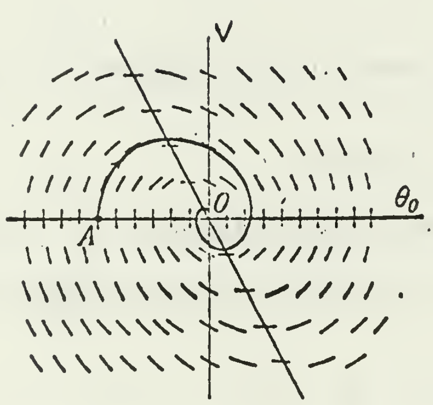


Fig. 2-5. Direction field obtained by Liénard's method.

C. DELTA METHOD

1. Method Description

Given

$$\ddot{x} + G(x,\dot{x},t) = 0 \tag{2-13}$$

where $(x,\ \dot{x},\ t)$ may be given analytically or graphically and may be a very complicated function.

Rewrite Eq.(2-13) to the standard delta form. This is done by adding and subtracting the term p^2x from Eq. (2-13).

$$\ddot{x} + [G(x,\ \dot{x},\ t) - p^2x] + p^2x = 0$$

Let

$$p^2\delta = [G(x,\ \dot{x},\ t),\ p^2x]$$

so that the linearity difference δ becomes

$$\delta = \frac{1}{p^2}[G(x,\ \dot{x},\ t) - p^2x] \tag{2-14}$$

Eq. (2-14) will make (2-13) appear in the standard δ form

$$\ddot{x} + p^2(x + \delta) = 0 \tag{2-15}$$

The geometric significance of (2-15) in a phase-plane representation is seen by introducing the phase-plane coordinates

$$x = x \quad \text{and} \quad \frac{\dot{x}}{p} = v.$$

Then

$$\ddot{x} = p\frac{dv}{dt} = p\frac{dv}{dt}\dot{x} = p^2v\frac{dv}{dx}$$

The standard second-order equation (2-15) then becomes a first-order differential equation in x, v, and δ, namely

$$v \frac{dv}{dx} + x + \delta = 0 \qquad (2\text{-}16)$$

$$\frac{dv}{dx} = -\frac{(x+\delta)}{v} \qquad (2\text{-}17)$$

Fig. 2-6 shows in this case that the slope of the line QP will be $\frac{x+\delta}{v}$; consequently the equation of the normal to QP will be identical with Eq. (2-17). The instantaneous phase trajectory is therefore a circular arc passing through P and with its center at $x = -\delta$ or at Q.

The next problem is therefore to relate the circular-arc segment beginning at P to the finite time interval Δt

Since

$$v = \frac{\dot{x}}{p}$$

then

$$\frac{dx}{dt} = pv$$

$$dt = \frac{dx}{pv} \qquad (2\text{-}18)$$

Fig. 2-7. Let the average radius of curvature of the step be given by $QP = \rho$.

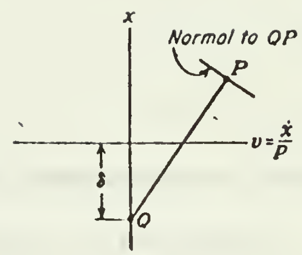


Fig. 2-6. Exact relationship.

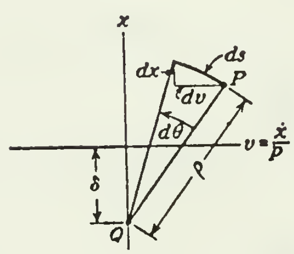


Fig. 2-7. Stepwise relationship for constant ρ.

Then

$$d\theta = \frac{ds}{\rho}$$

$$= \frac{\sqrt{dx^2 + dv^2}}{\sqrt{v^2 + (x+\delta)^2}}$$

$$= \frac{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}\, dx}{\sqrt{1 + \left[\frac{(x+\delta)}{v}\right]^2}\, v}$$

$$= \frac{dx}{v}$$

$$\therefore \ dt = \frac{1}{p}\, d\theta \qquad (2\text{-}19)$$

Consequently, an integration will give

$$t - t_o = \frac{1}{p}(\theta - \theta_o)$$

$$p\,\Delta t = \Delta\theta \qquad (2\text{-}20)$$

Eq. (2-20) shows, in regard to the angle subtended by the circular arc described by the phase point during the finite time interval Δt.

A graphical integration of the differential equation in the standard delta form can then be effected by a successive construction of circular arc segments whose centers are located on the x-axis at various step value of $x = -\delta$ (see Fig. 2-8a).

(a) Change with positive δ.   (b) Change with negative δ.

Fig. 2-8a.

If x had been taken horizontal and positive to the right with v positive vertically, positive time would have been represented by a clockwise angular variation. (see Fig. 2-8b).

The definition of δ given by Eq. (2-14) transforms the original Eq. (2-13) into the standard delta form. Curves representing δ in terms of the phase coordinates x and/or $\frac{\dot{x}}{p}$ will be superimposed on the phase-plane so as to permit choice of a suitable average value of δ for the construction of the circular-arc segments making up the phase trajectory.

2. Example (II-B-2), Mass on Non-linear Spring

The motion of a constant mass attached to a non-linear spring which becomes relatively stiffer with increased deflection is described by an equation of the form

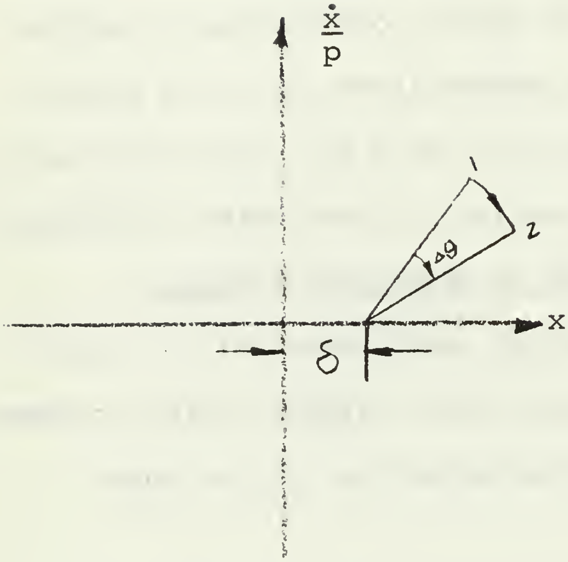
$$\frac{d^2x}{dt^2} + 25(1 + 0.1\ x^2)x = 0$$

Find a phase-plane solution curve for this equation by δ method, with the initial condition that at t = 0, x = 3 and $\frac{dx}{dt} = 0$.

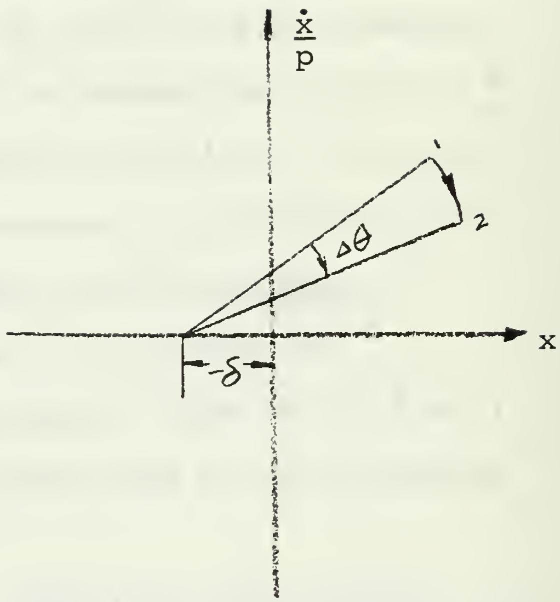
Comparing to the standard delta-form (Eq.(2-15))

$$p^2(x+\delta) = 25(1+0.1\ x^2)x$$

$$\begin{cases} p^2 = 25 \\ \delta = 0.1x^3 \end{cases}$$



(a) Change with positive δ.



(b) Change with negative δ.

Fig. 2-8b.

First draw the δ curve. Phase trajectory construction proceeds as shown in the figure. The initial point is $x(0) = 3$ and $v(0) = 0$, as the first step. It is assumed that x decreases to the value 2.8 or $\Delta x^{(1)} = -0.2$ as shown. The average value of x during this interval is $x_{av}^{(1)} = 2.9$, for which the average value of δ can be read directly as $\delta_{av}^{(1)} = 2.4$ approximately. Thus the center of the first circular arc is located at point $x = -\delta_{av}^{(1)} = -2.4$ and $v = 0$; the radius is $R = 5.4$ and with this radius an arc is drawn to the point where $x = 2.8$, indicated by the small dot marked $x(0) + \Delta x^{(1)}$ in the figure. By continuing in this manner, the entire solution curve can be built up as a sequence of circular arcs. (Since the original equation involves no damping, the solution is a closed curve representing a periodic oscillation. (See Fig. 2-9).

3. Discussion

a. The delta method is limited to construction of phase trajectories for second-order differential equations.

b. In starting a solution by the isocline method, the entire plane must be fitted with line segments fixing the slope of a solution curve. If only a single solution curve is needed, only a few of these line segments are actually put to use. Considerable simplification would result if only information related directly to the desired solution curve were used.

The delta (δ) method leads more directly to the desired solution.

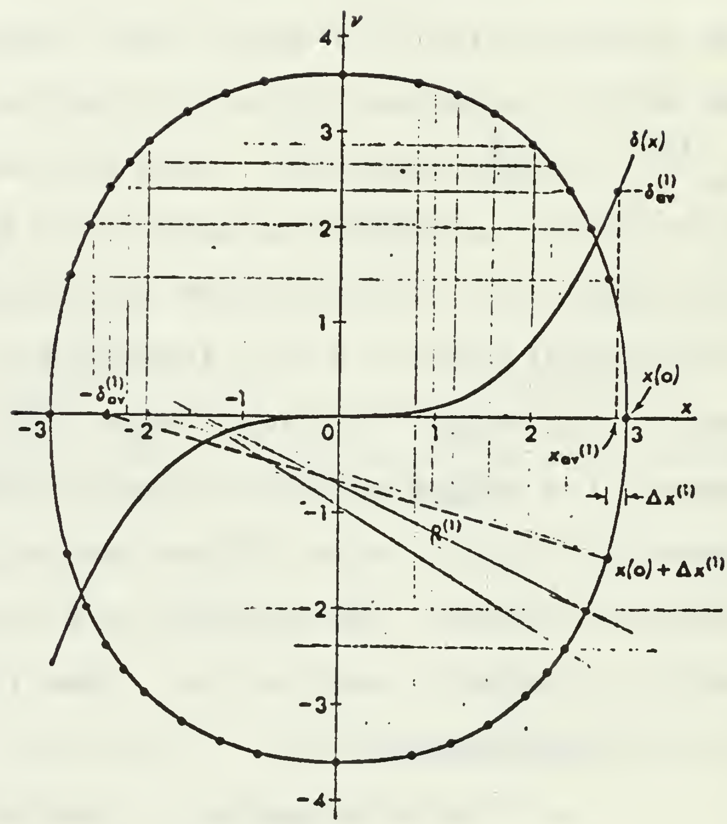


Fig. 2-9. Phase-plane diagram constructed by δ method. The curve for non-linear function δ(x) and the final solution curve are shown. Dots indicate junction of successive arcs used in building up solution.

c. The construction of phase trajectories will be carried out for various magnitudes of steps. So the accuracy may depend on different amounts of constructional work.

4. Notes

a. δ may be either positive or negative. The following δ curves for five systems may help to understand how plots of curves may be superposed on the phase-plane.

(1) The physical pendulum (See Fig. 2-10)

$$\ddot{x} + p^2 \sin x = 0$$

$$\delta = \frac{1}{p^2} [G(x) - p^2 x] = \sin x - x$$

(2) A system with restoration proportional to the cube of the displacement (See Fig. 2-11)

$$\ddot{x} + \beta x^3 = 0$$

$$\delta = \frac{\beta}{p^2} x^3 - x$$

(3) A linear restoration system with quadratic velocity damping (See Fig. 2-12)

$$\ddot{x} + q \, |\dot{x}| \, \dot{x} + p^2 x = 0$$

$$\delta = \frac{1}{p^2} [G(x,\dot{x}) - p^2 x] = \frac{q}{p^2} \, |\dot{x}| x = q \left| \frac{\dot{x}}{p} \right| \frac{\dot{x}}{p}$$

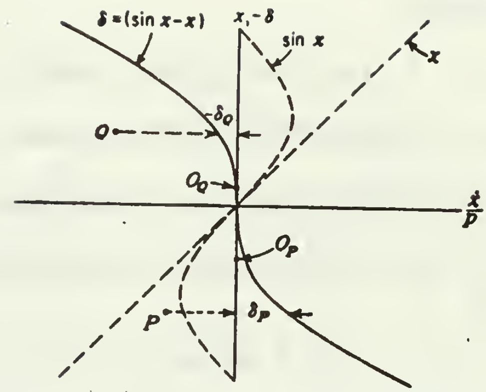


Fig. 2-10. The operative displacement δ for the physical pendulum plotted in the phase-plane. Centers O_P and O_Q correspond to points P and Q.

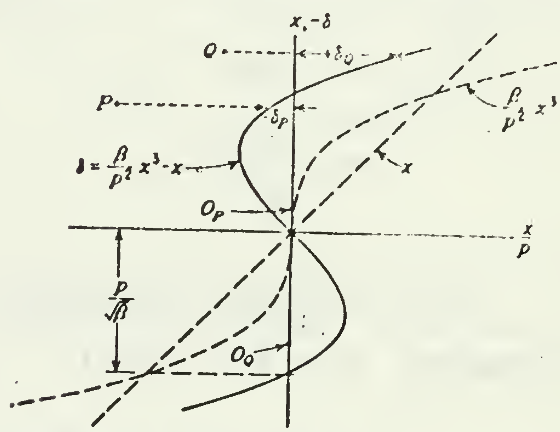


Fig. 2-11. The operative displacement δ for a system with restoration proportional to the cube of the displacement. Centers O_P and O_Q correspond to points P and Q.

(4) A system with positive linear and with negative quadratic restoration in x (See Fig. 2-13)

$$\ddot{x} + p^2 x - a\,|x|\,x = 0$$

$$\delta = \frac{1}{p^2}(p^2 x - a\,|x|\,x - p^2 x) = -\frac{a}{p^2}\,|x|\,x$$

(5) A system acted upon by nth-power velocity damping and restored by first and mth-power displacement restoration (See Fig. 2-14)

$$\ddot{x} + \alpha\,|\dot{x}^{n-1}|\,\dot{x} + p^2 x + \beta\,|x^{m-1}|\,x = 0$$

$$\delta = \frac{\alpha}{p^{2-n}}\left|\left(\frac{\dot{x}}{p}\right)^{n-1}\right|\frac{\dot{x}}{p} + \frac{\beta}{p^2}|x^{m-1}|x$$

$$= \delta_1 + \delta_2$$

(b) The δ curve of a system indicates the system's type and degree of non-linearity. Thus, if a system is linear, δ will be zero unless an artificial parameter has been introduced deliberately.

D. SZEGO METHOD

1. Method Description

Consider the second-order system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x,y) \end{cases} \tag{2-21}$$

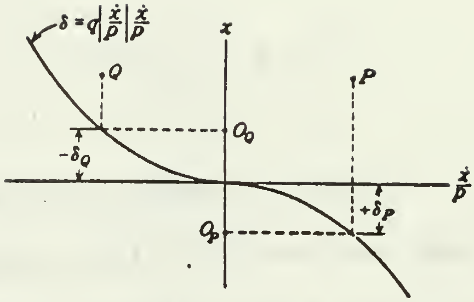


Fig. 2-12. The operative displacement δ for a system with quadratic velocity damping. Centers O_P and O_Q correspond to points P and Q.

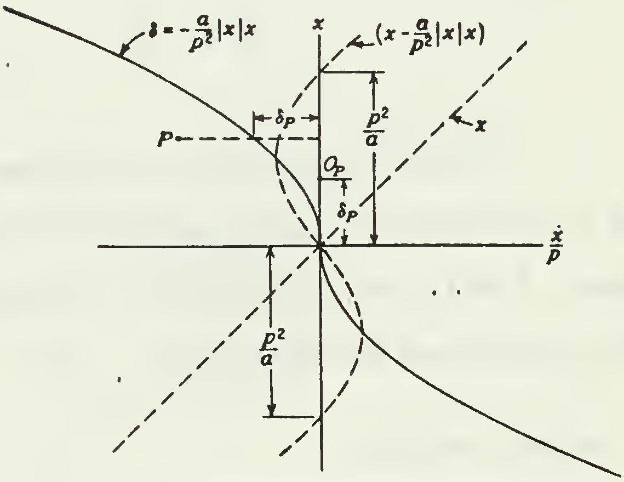


Fig. 2-13. The operative displacement δ for a system with positive linear and negative quadratoc restoration in displacement. Center O_P corresponds to point P.

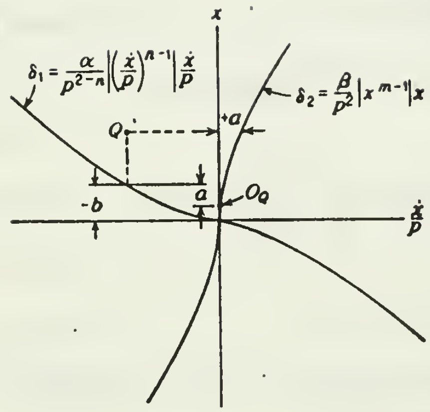


Fig. 2-14. The two operative displacements δ_1 and δ_2 for a system with damping proportional to the nth power of the velocity and with restoration proportional to the first and the mth power of the displacement. Center O_Q corresponds to point Q.

It is desired to identify the slope of the trajectory of this system at every point of the phase space. We consider the topographical system; that is a system of non-intersecting differential curves, which are symmetric with respect to the origin.

$$v = \phi(x,y) \tag{2-22}$$

In connection with the function (2-22), construct the curves defined by the equation

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} = 0 \tag{2-23}$$

and Eq. (2-21).

These curves we called contact curves. Substituting Eq. (2-21) into (2-23)

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = -\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{f(x,y)}{y} \tag{2-24}$$

At every point of the curve (2-24), the corresponding trajectory of the system (2-21) has the same slope as the curve of the family (2-22) at the same point; furthermore, the curves, representing in the phase-plane the solutions with odd-order of multiplicity of Eq. (2-23) divide the phase-plane into regions in which $\frac{dv}{dt}$ has fixed sign. On the contrary, $\frac{dv}{dt}$ has the same sign on both sides of the curves, representing the solution with even order of multiplicity of Eq. (2-23). Of course, in the latter case the slope of the trajectories at every point on these curves is the same as the slope of the family (2-22).

In regions where $\frac{dv}{dt}$ is negative, the representative point of the system (2-21) describes a trajectory in the phase-plane which crosses the curves of the family curves (2-22) in the direction of decreasing v.

In the regions in which $\frac{dv}{dt}$ is positive the trajectories cross the curves of the family (2-22) in the direction of increasing v.

Choosing different families of curves (2-22) and plotting the corresponding curves $\frac{dv}{dt} = 0$ in the phase-plane, provides enough information to sketch the trajectories.

If the scalar function, $v = \phi(x,y)$ changes sign in the phase-plane, i.e., if the equation $v = \phi(x,y)$ has some real non-zero solution with odd-order of multiplicity, then the curves representing these solutions must also be considered as boundaries between regions with different sign of $\frac{dv}{dt}$. This must be done in order to take into account the variation in sign of $\frac{dv}{dt}$, due to variation in the sign of the curves of the topographical system (2-22). Notice that the trajectories of the system (2-21) on the curves $v = 0$ have not the same slope as curves of the family (2-22).

Consider the particular topographical systems:

$$v = x \tag{2-25}$$

for which

$$\frac{dv}{dt} = y \tag{2-26}$$

and

$$v_1 = y \tag{2-27}$$

for which

$$\frac{dv_1}{dt} = f(x,y) \tag{2-28}$$

The subsequent families of curves are now considered as topographical systems.

$$v_2 = x^2 + y^2 \tag{2-29}$$

$$v_3 = x^2 - y^2 \tag{2-30}$$

$$v_4 = xy \tag{2-31}$$

Using $v_1 = y$

when $v_1 = 0$

for $y = 0$ (2-32)

and from

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial x}\frac{dx}{dt} + \frac{\partial v_1}{\partial y}\frac{dy}{dt} = \frac{dy}{dt} = 0$$

then

$$f(x,y) = 0 \tag{2-33}$$

When considering $v_2 = x^2 + y^2$; it is seen that $v_2 = 0$ for $x = y = 0$ and

$$\frac{dv_2}{dt} = 0$$

from

$$\frac{dv_2}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$= 2xy + 2y\,f(x,y) = 0$$

then

$$x + f(x,y) = 0$$

and

$$y = 0 \tag{2-34}$$

For

$$v_3 = x^2 - y^2$$

$$v_3 = 0 \quad \text{for} \quad x = y$$

$$\text{and} \quad x = -y \tag{2-35}$$

Also

$$\frac{dv_3}{dt} = 0 \, ,$$

But

$$\frac{dv_3}{dt} = \frac{\partial v_3}{\partial x} \frac{dx}{dt} + \frac{\partial v_3}{\partial y} \frac{dy}{dt}$$

$$= 2x \frac{dx}{dt} - 2y \frac{dy}{dt}$$

$$= 2xy - 2yf(x,y)$$

$$= 0$$

Hence

$$x = f(x,y)$$

and

$$y = 0 \tag{2-36}$$

Finally, considering $v_4 = xy$, gives

$$v_4 = 0$$

for $x = 0$ and $y = 0$ (2-37)

and

$$\frac{dv_4}{dt} = 0 \; ,$$

But

$$\frac{dv_4}{dt} = \frac{\partial v_4}{\partial x} \frac{dx}{dt} + \frac{\partial v_4}{\partial y} \frac{dy}{dt}$$

$$= y \frac{dx}{dt} + x \frac{dy}{dt}$$

$$= y \cdot y + x f(x,y)$$

$$= 0$$

Hence

$$y^2 = -xf(x,y) \tag{2-38}$$

2. Example (II-D-2)

Consider Van der Pol's equation:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \epsilon(1-x^2)y - x \end{cases} \tag{2-39}$$

(a) $\frac{dv_1}{dt} = 0$ for $y = \frac{x}{1 - x^2}$

(b) $\frac{dv_2}{dt} = 0$ for $y^2 = 0$

$x = \pm 1$

There is no variation in the sign of $\frac{dv_2}{dt}$ between the two sides of the curve $y^2 = 0$.

(c) $\frac{dv_3}{dt} = 0$ for $y = 0$

$y = \frac{2x}{1 - x^2}$

(d) $\frac{dv_4}{dt} = 0$ for $y = \frac{x}{2} [(x^2-1) \pm \sqrt{(x^2-1)^2 + 4}]$

The next step is to plot the curves (a), (b), (c) and (d), (see Figs. 2-15, 2-16, 2-17 and 2-18).

In order to use the method, it is necessary to superimpose the curves drawn in Figs. 2-15, 2-16, 2-17, and 2-18 as is shown in Fig. 2-19.

The best procedure now is to check the sign of $\frac{dv_i}{dt}$ ($v = 1,2,3,4$) numerically at one point of the phase-plane and utilize the result for identifying the sign of $\frac{dv_i}{dt}$ ($i = 1,2,3,4$) in the different regions in which curves $\frac{dv_i}{dt} = 0$ and $v = 0 (i = 1,2,3,4)$ divide the phase-plane. Since

$$0 < v_2 < +\infty$$

$$-\infty < v_1 < +\infty$$

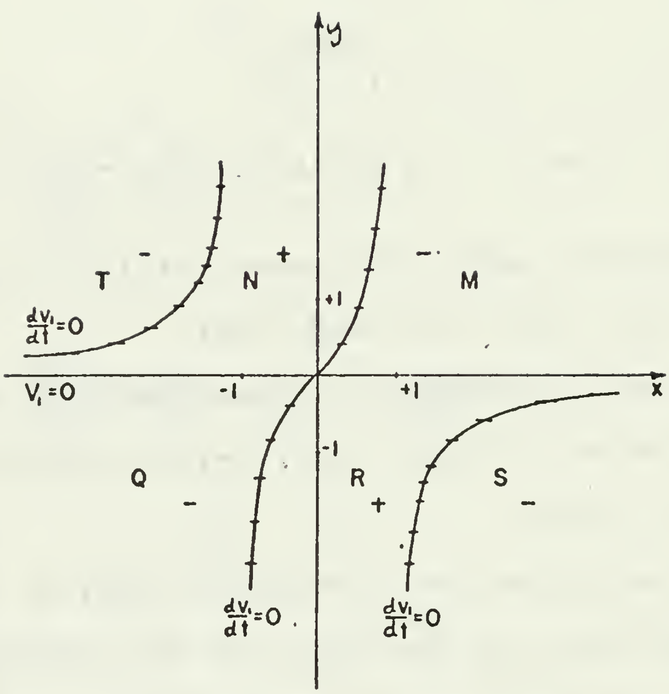
$$-\infty < v_3 < +\infty$$

$$-\infty < v_4 < +\infty$$

It is necessary, before checking the sign of $\frac{dv_1}{dt}$, $\frac{dv_3}{dt}$, and $\frac{dv_4}{dt}$ at one point of the phase-plane, to identify the sign of v_1, v_3 and v_4 at that point.

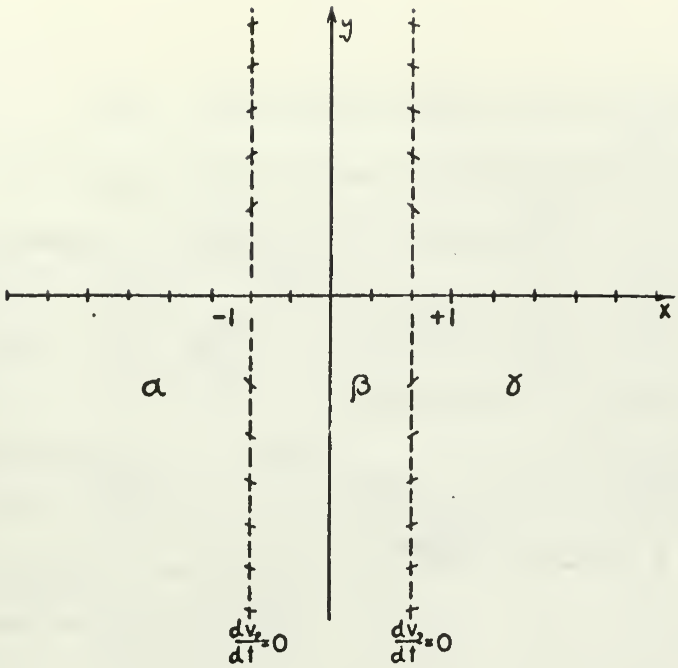
$v_1 = 0$ for $y = 0$

$\frac{dv_1}{dt} = 0$ for $\frac{x}{1-x^2}$



generate regions M,N,T,Q,R and S

Fig. 2-15. Curves $v_1 = 0$, and $\frac{dv_1}{dt}$



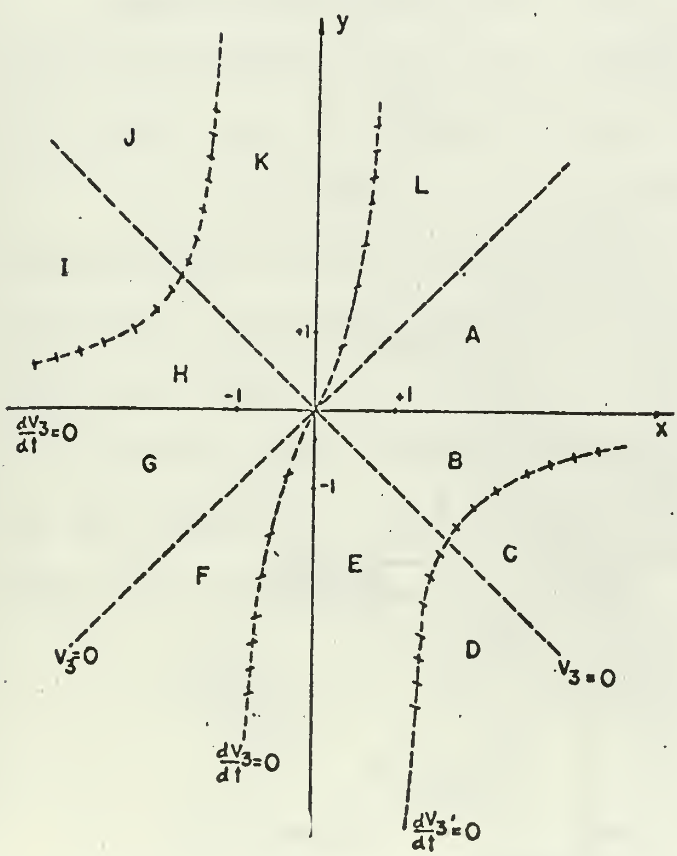
$$\frac{dv_2}{dt} = 0 \quad \text{for}$$

$$y^2 = 0$$

$$x = \pm 1$$

generate regions α, β and γ

Fig. 2-16. Curves $v_2 = 0$ and $\frac{dv_2}{dt} = 0$.

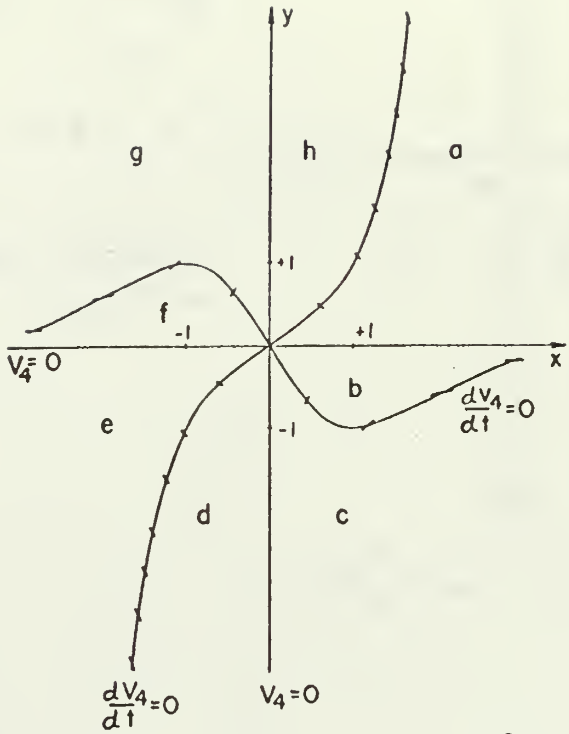


$v_3 = 0$ for $x = y$

$\frac{dv_3}{dt} = 0$ for $y = \frac{2x}{1-x^2}$

generate regions A, B, C, D, E, F, G, H, I, J, and K

Fig. 2-17. Curves $v_3 = 0$ and $\frac{dv_3}{dt} = 0$.



$v_4 = 0$

for $x = 0$, $y = 0$

$\frac{dv_4}{dt} = 0$

for $y = \frac{x}{2}\left[(x^2-1) \pm \sqrt{(x^2-1)^2+4}\right]$

generate regions
a,b,c,d,e,f,g, and h

Fig. 2-18. Curves $v_u = 0$ and $\frac{dv_u}{dt} = 0$.

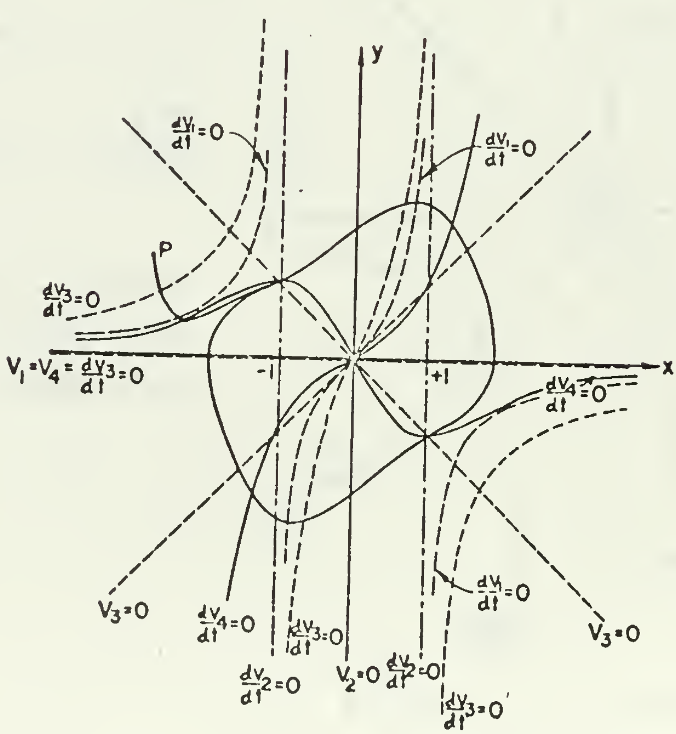


Fig. 2-19. Example (II-D-2)

Some information about the slope of the trajectories at every point of the phase-plane is thus provided. For example, consider the point P(-3; 1.5) in Fig. 2-20),

from $\left.\frac{dv_2}{dt}\right|_P < 0$, requires that the trajectory move into the region A'F'D'B'

from $\left.\frac{dv_3}{dt}\right|_P > 0$, the trajectory moves from P into the region E'C'A'F'

from $\left.\frac{dv_4}{dt}\right|_P < 0$, requires that the trajectory move into the region C'A'F'D'.

These three conditions, taken simultaneously, define the angle A'PF' in which the trajectory which originates at P must lie.

Since the slope of the trajectories is known exactly at the points on the curves $\frac{dv_i}{dt} = 0$, it is possible to draw the trajectories starting from P with good accuracy. (See Fig. 2-19)

Starting for example from point P(-3,1.5) contained in the region of fixed sign of $\frac{dv_i}{dt}$, namely T, α, I and G. (See Fig. 2-19)

$$\left.\frac{dv_1}{dt}\right|_P < 0$$

$$\left.\frac{dv_2}{dt}\right|_P < 0$$

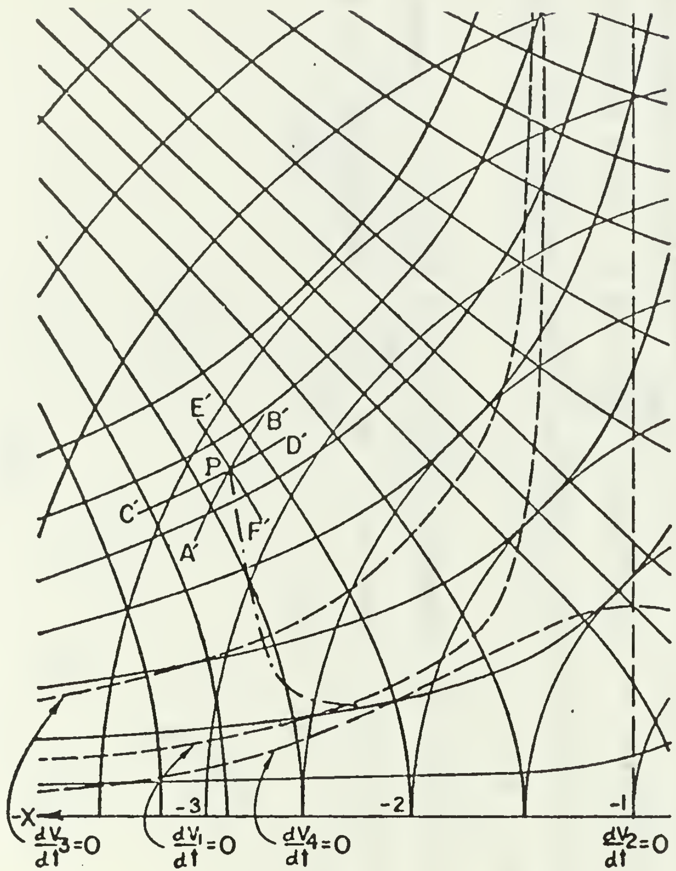


Fig. 2-20. Exact generation of trajectory from point P.

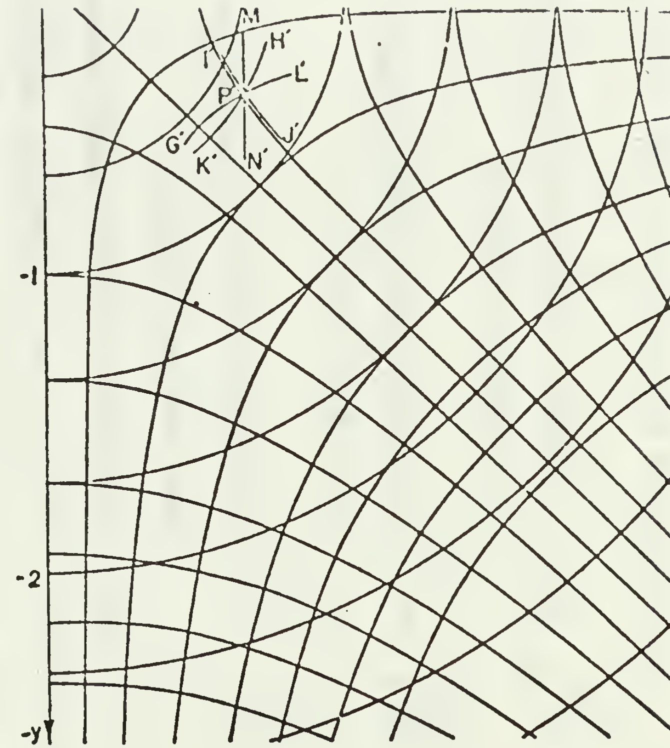


Fig. 2-21. Exact generation of trajectory from the point P' by using v-function, v = x.

$$\left.\frac{dv_3}{dt}\right|_P > 0$$

$$\left.\frac{dv_4}{dt}\right|_P < 0$$

NOTE:

$$\frac{dv_1}{dt} < 0 \quad \text{in} \quad T,\ Q,\ M \text{ and } S$$

$$\frac{dv_1}{dt} > 0 \quad \text{in} \quad N \text{ and } R$$

$$\frac{dv_2}{dt} < 0 \quad \text{in} \quad \alpha \text{ and } \gamma$$

$$\frac{dv_2}{dt} > 0 \quad \text{in} \quad \beta$$

$$\frac{dv_3}{dt} < 0 \quad \text{in} \quad B,\ D,\ F,\ H,\ J \text{ and } L$$

$$\frac{dv_3}{dt} > 0 \quad \text{in} \quad A,\ C,\ E,\ G,\ I \text{ and } K$$

$$\frac{dv_4}{dt} > 0 \quad \text{in} \quad b,\ d,\ f \text{ and } k$$

$$\frac{dv_4}{dt} < 0 \quad \text{in} \quad a,\ c,\ e \text{ and } g$$

3. <u>Discussion</u>

a. This method is a simple and useful method for drawing trajectories. Also it is essentially a generalization of the isocline method.

b. Accuracy can be improved by considering further topographical systems.

c. It is worthwhile to notice that the isocline method is a particular case of this method. And it corresponds to the particular topographical system:

$$v = a_1 x + a_2 y$$

$$(-\infty \leq a_i \leq +\infty \,; i = 1,2)$$

NOTE:

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\ \frac{dx}{dt} + \frac{\partial v}{\partial y}\ \frac{dy}{dt}$$

$$= a_1 \frac{dx}{dt} + a_2 \frac{dy}{dt}$$

$$= 0$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{a_1}{a_2}$$

E. MURTHY'S NEW APPROACH TO THE PLOTTING OF PHASE-PLANE TRAJECTORIES

1. Method-Description

This method is a new and simple approach for the isocline plotting of second-order time-implicit non-linear differential equations given by $\ddot{x} = F(x,x')$.

a. Isoclines of the equation $x''+f(x')+g(x) = 0$:

$$x'' + f(x') + g(x) = 0 \qquad \text{(2-40a)}$$

Substituting $Mx' = x''$ into (2-24a)

where $M = \frac{dx'}{dx}$; gives

$$Mx' + f(x') + g(x) = 0 \tag{2-40b}$$

(1) Plot from the given differential equation (2-40a) the curves of x'' vs x' on the $x''x'$ plane for constant (different) values of x. This is a simple matter since it amounts to shifting the curves of $x'' = f(x')$ vertically for different values of x.

(2) Draw a number of straight lines $x'' = Mx'$ through the origin (corresponding to the different values pf M) on the $x''x'$ plane; this can be done quickly.

(3) With any particular straight line in view, say for $M = M_1$, determine all possible points of intersections of this straight line with the curves obtained from (a). Thus a number of values of x and x' can be obtained corresponding to $M = M_1$, which when plotted on the $x'x$ plane and joined together represent the isocline for $M = M_1$. The procedure is repeated for $M = M_2$, M_3 ... until a fair number of isocline curves are obtained on the phase plane. The respective slope lines are marked on these isoclines. Joining these slope lines gives the different trajectories or the entire phase portrait for Eq. (2-40a).

(b) Isoclines for the equation

$$x'' + f(x')h(x) + g(x) = 0 \tag{2-41}$$

The above procedure can still be applied except that the curves of $x'' = f(x')$ are now multiplied by a scale factor [due to the $h(x)$ term] corresponding to different values of x.

(c) Notes

(1) This method is more convenient in determining the solution curves for the following differential equations:

$$\begin{aligned} x'' + x'h(x) + g(x) &= 0 \\ x'' + x' \quad + g(x) &= 0 \\ x'' + xx' \quad + g(x) &= 0 \end{aligned} \tag{2-42}$$

when compared with existing methods.

(2) This method can also be used for the following nonlinear differential equations:

$$\begin{aligned} h(x)x'' + K(x)\ f(x') + g(x) &= 0 \\ h(x')x'' + K(x)\ f(x') + g(x) &= 0 \\ x'' &= F(x, x') \end{aligned} \tag{2-43}$$

(3) The Lienard method is a step-by-step construction for drawing any particular trajectory corresponding to a given set of initial conditions. The present method determines the entire phase portrait indicating the behavior for all initial conditions.

(4) Variations of the nonlinear restoring-force term $g(x)$ can easily be studied.

(5) The method is independent of the nature of the functions f,g,h,k, i.e., whether logarithmic, algebraic, or transcendental.

(6) For nonlinear differential equations of the type $x'' = f(x)$, the x''-x' curves become horizontal straight lines for constant values of x. The intersections of these straight lines with $x'' = Mx'$ lines give the points on the phase plane for any chosen isocline slope.

(7) The isoclines corresponding to $M = 0$ and $M = \infty$ can generally be plotted quickly.

(8) Sometimes an x''-x' curve (for a constant x) and an $x'' = Mx'$ line may run so close to one another that the isocline curve on the phase plane is nearly vertical in that range of x'.

2. Example

Consider a system with nonlinear damping and restoring forces described by $x'' + x'^2 + x^2 = 0$.

(a) Draw x''-x' curves (see Fig. 2-22).

(b) Draw $Mx' = x'$ corresponding to various values of M (a number of straight lines through the origin. (See Fig. 2-22)

(c) Draw isoclines and phase trajectories for

$$x'' + x'^2 + x^2 = 0.$$

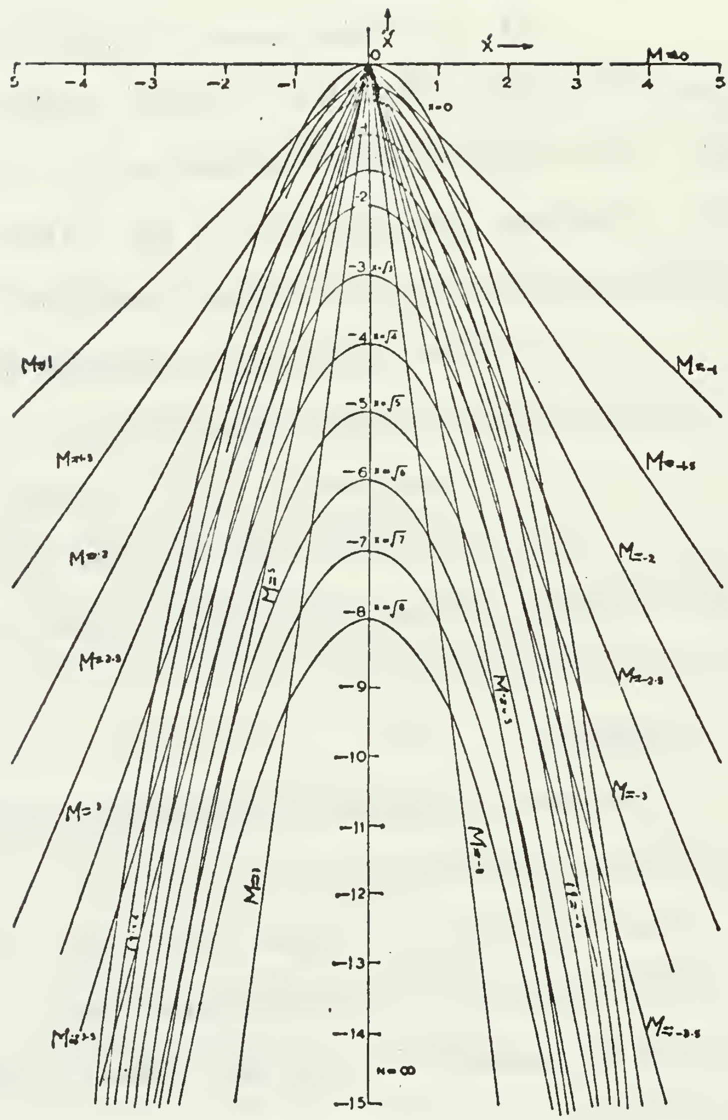


Fig. 2-22. x"/x' curves and x" = Mx' lines for the equation $x'' + x'^2 + x^2 = 0$.

Considering the line for, say $M = +5$, the points of intersections are determined as

$$(x = 0 \ , \ x' = 0)$$

$$(x = 2 \ , \ x' = -1)$$

$$(x = \sqrt{3} \ , \ x' = -7)$$

$$(x = \sqrt{5} \ , \ x' = -1.4)$$

$$(x = \sqrt{6} \ , \ x' = -2.0)$$

$$(x = \sqrt{6} \ , \ x' = -2.5)$$

These points are marked on the phase-plane, joined and the slope lines are marked on it (Fig. 2-23). Similarly, different isoclines are obtained for different values of M. The trajectories are then drawn as shown in Fig. 2-23.

3. Discussion

(a) This is a simple, all graphical, useful and general method to obtain the phase-plane trajectories corresponding to all pertinent initial conditions of second-order autonomous systems (which are not amenable to the usual isocline method) by making use of the isocline method. It reduces the labor of calculating isoclines for many non-linear differential equations and thus makes the isocline method a much more useful tool.

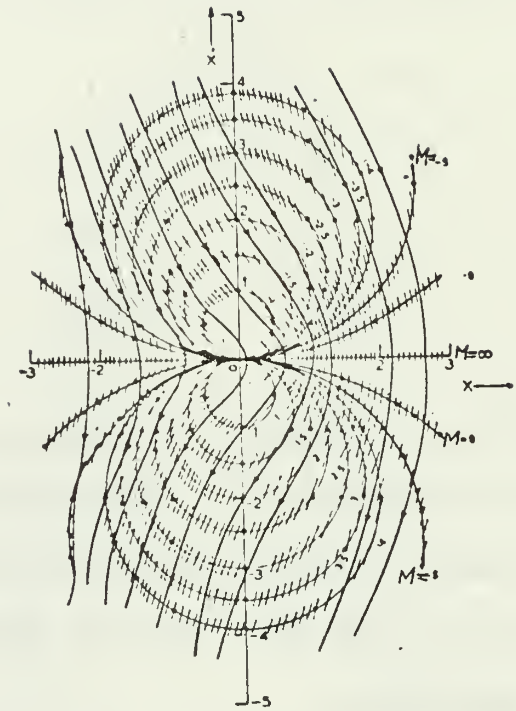


Fig. 2-23. Isoclines and phase trajectories for $x'' + x'^2 + x^2 = 0$.

F. THE ACCELERATION-PLANE METHOD

1. Method-Description

It is based on the fundamental equation

$$\frac{dv}{dx} = \frac{w}{v} , \qquad (2\text{-}44)$$

where $w = w(t, v, x)$ or the acceleration is a function of t, v and x. For the sake of simplicity, assume that a constant driving force is applied.

Thus $$\frac{dv}{dx} = \frac{w(v,x)}{v} \qquad (2\text{-}45)$$

The method calls for preliminary plots of w vs x for constant values of v. This auxiliary w-x plot for constant values of v is not a trajectory. It characterizes the system just as the original differential equation characterizes the physical condition of the system.

This method is applicable to both linear and non-linear problems. In the w-x plot, though constant driving functions can be included, the time-varying forcing or driving functions must be added for different values of time.

2. Example $x'' + 0.5x' + x = 0$.

For $v = x' = 0$, then $x'' = -x$, and the $x'' - x$ plot or w-x plot is a straight line with a slope -1 . For $v = x' = 1$, then $x'' = -x - 0.5$, so this straight line is displaced from the first line by -0.5.

In Fig. 2-24(a) six points of construction are shown for the v-x trajectory, starting from P_o where $t = 0$, $x_o = 2.5$ and $v_o = 0$.

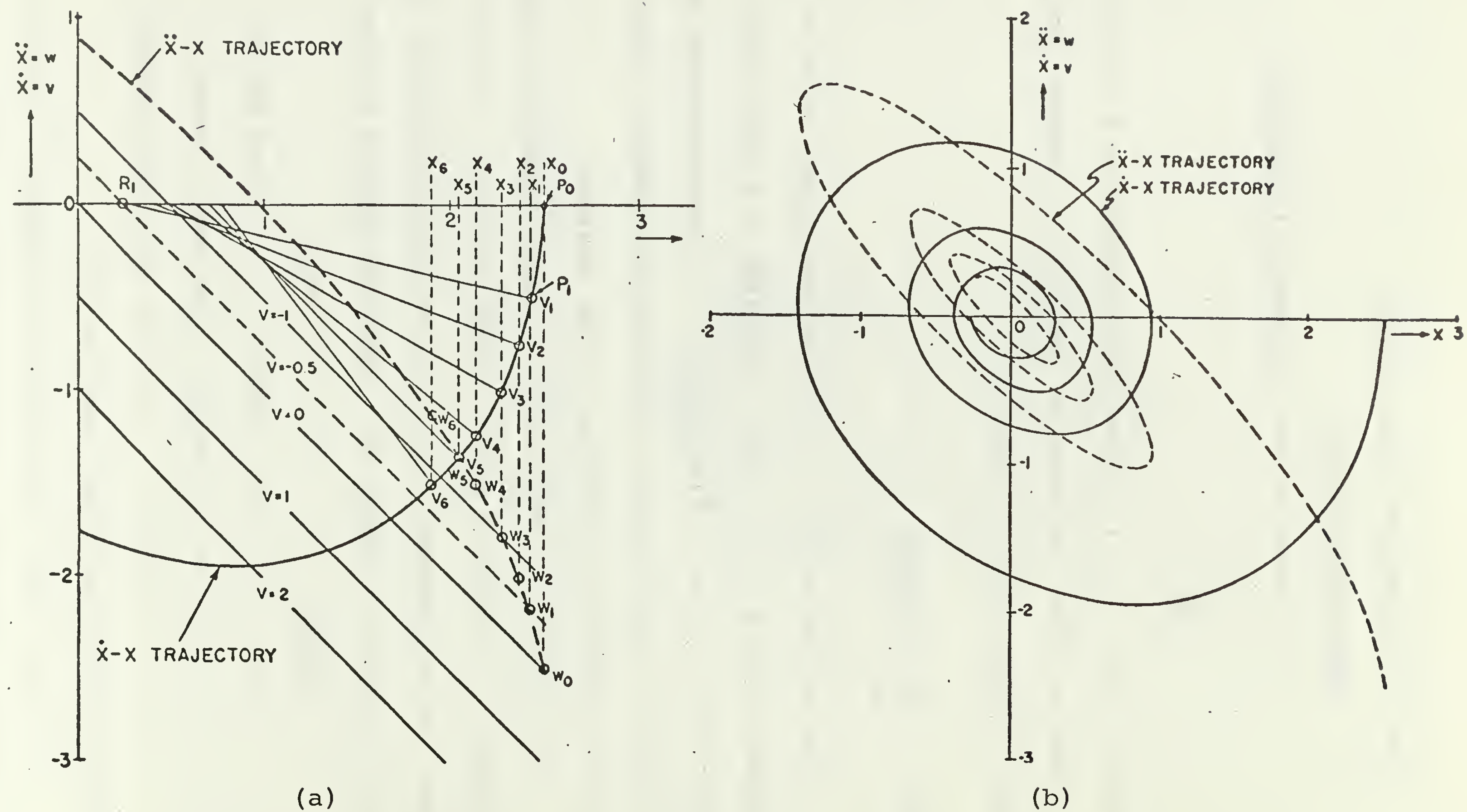


Fig. 2-24. Construction of trajectories by acceleration-plane method.

Taking x_1 as the center, and starting from w_1, an arc can be drawn in the clockwise direction so that the arc strikes R_1 on the horizontal axis. From R_1 draw a line to P_1 where $x = x_1$ and $v = v_1$; the perpendicular line to P_1R_1 gives the correct slope for the v-x trajectory at P_1. If the perpendicular line gives a slope that agrees fairly well with the assumed slope, point P_1 can be considered as the desired point which falls on the v-x trajectory.

Slope of perpendicular line $= \dfrac{w_1}{v_1}$.

If the assumed slope does not agree with the slope of the perpendicular line obtained from the above construction, a new slope should be assumed and the above procedure repeated until the assumed slope agrees with the slope of the perpendicular line (see Fig. 2-25). On Fig. 2-24(a), P_1 has the value $x = x_1$ and $v = v_1$ and v_1 is chosen as -0.5. Thus if one prolongs the vertical line from x_1 downward through P_1, this line should intersect with the $v = -(\frac{1}{2})$ line in w-x plot (shown in dot lines) at a point w_1. By connecting the points w_o, w_1, w_2 ... etc., the w-x trajectory is constructed at the same time (see Fig. 2-24(b)).

3. Discussion

This method is applicable to both linear and nonlinear problems. One draws the w-x trajectory first, then gets the v-x phase-plane trajectory. Each time only one trajectory is obtained, so this is a useful method.

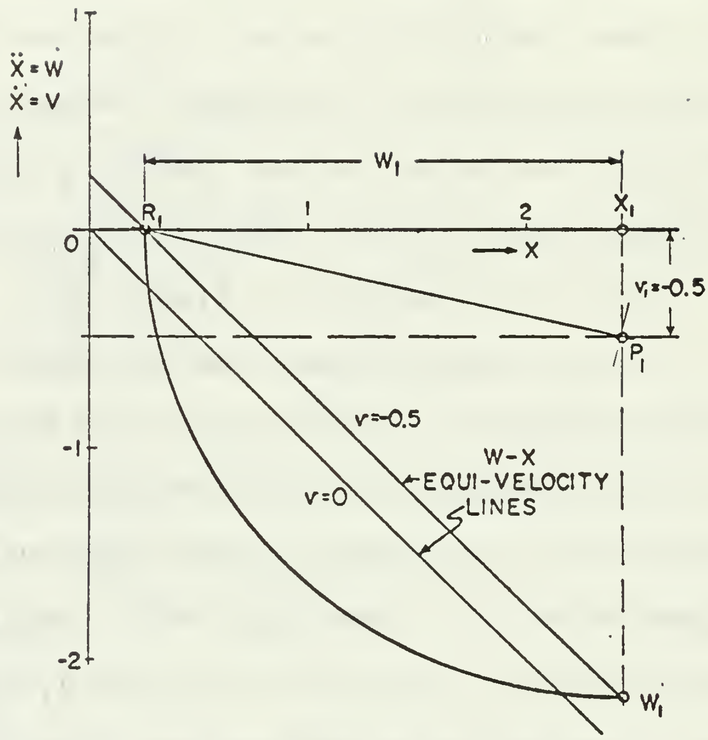


Fig. 2-25. Acceleration-plane method.

G. DEEKSHATULU'S METHOD

This method suggests the use of simple transformations for second-order non-linear differential equations to effect rapid plotting of the phase-plane trajectories.

1. Method-Description

a. Transformations for second-order autonomous systems

A general second-order non-linear autonomous system is governed by a differential equation of the form

$$\ddot{x} = f(x,\dot{x}) \tag{2-46}$$

Let

$$y = \frac{dx}{dt}$$

then

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= M \cdot \frac{dx}{dt}$$

Change Eq. (2-46) to

$$M\dot{x} = f(x,\dot{x}) \tag{2-47}$$

To draw the isocline for an assumed value of M, equal to M_1, say, $\dot{x}$ has to be found as a function of x and M_1 and this in many cases is not possible; thus the only attempt made is to find the points of intersections of such an isocline with a given set of straight lines or with simple standard curves. Consequently, on the phase-plane, if one considers the intersections of the isocline for slope $M = M_1$ and the set of straight lines $\dot{x} = Kx (K = \pm 1, K = \pm 2, \ldots)$, Eq. (2-47) becomes

$$M_1 Kx = f(x, Kx)$$

or

$$M_2 = F(x, K) \tag{2-48}$$

For a specified K, the solution of Eq. (2-48) (algebraic equation with constant coefficients) for real roots of x gives the points of intersection. But it is easier to find the values of the slope M (for various values of x) on the lines $\dot{x} = Kx$ than it is to try to obtain the isocline curve for a given value of the slope. Even under this transformation $\dot{x} = Kx$, every quadrant of the phase-plane is uniquely defined. It is advisable to find any symmetry present in the phase portrait for a given differential equation since, if it exists, it reduces labor and is tested as follows:

(1) <u>Symmetry about the x-axis</u>

If for K = positive (negative) K_1 and x = positive (negative) x_1, the slope $M = M_1$, then for K = negative (positive) K_1, and x = positive (negative) x_1, the slope $M = -M_1$.

(2) <u>Symmetry about the $\dot{x}$-axis</u>

If for K = positive (negative) K_1 and x = positive (negative) x_1, the slope $M = M_2$, then for K = negative (positive) K_1, and x = negative (positive) x_1, the slope $M = -M_2$.

The procedure for determining M_1 on the phase-plane for any chosen value K_1 and x_1 can be viewed in the phase space (with x, $\dot{x}$, and $\ddot{x}$ as coordinate axes). Determine the curve of intersection of the surface defined by Eq. (2-46) with the plane $\dot{x} = K_1 x$, then determine the simple point of intersection P of this curve with the plane $x = x_1$. The slope of the plane passing through the x-axis and the point P with x-$\dot{x}$ plane determines the value of M_1. Similarly, other values of M are obtained by taking different values of x and the procedure is repeated for each value of K.

Other transformations such as $\dot{x} = Kx^2$, $\dot{x} = Kx^3$, x or $x = K$, etc., can also be considered.

This procedure is general, systematic, and comparatively simple when compared with the method of determining slope lines arbitrarily at points for the phase portrait construction.

b. New Planes for Certain Second-order Time-Varying Systems.

Time-varying elements occurring in physical systems; for example, the variable mass of a guided missile can be though of as function of time since its weight decreases as the fuel is consumed.

On the phase-plane, solution of this kind of problem makes use of step-by-step procedures by considering simultaneously the $\dot{x}$-x and x-t (solution) planes. Professor Deekshatulu suggests some new planes whose coordinate axes are explicit functions of time and which effect an easy and rapid solution of at least certain classes of second-order non-linear time-varying systems not easily amenable otherwise by the phase-plane methods.

c. Suggestions for Solving Certain Higher-order Systems

Assuming non-linear transformations of the form

$$\xi = \dot{e} + af(e) \quad \text{or} \quad \eta = \ddot{e} + bf(e) + a\dot{e} \qquad (2\text{-}49)$$

third and fourth-order time-varying systems described by the following equations:

$$\left\{\begin{array}{l} t^2\ddot{\xi} + \dot{\xi} + f(\xi) = 0 \\ 2t\ddot{\xi} + \dot{\xi} + f(\xi) = 0 \\ t^2\ddot{\eta} + t\dot{\eta} + f(\eta) = 0 \\ 2t\ddot{\eta} + \dot{\eta} + f(\eta) = 0 \end{array}\right. \qquad (2\text{-}30)$$

can first be solved by the method described and then be solved as a first-order or second-order nonautonomous system. Note that the equation of the form $\dot{e} + af(e) = h(t)$ can be solved on the e-t (solution) plane.

2. Examples

(Example 1) Consider a non-linear system with dissipation proportional to $\dot{x}^2$, given by $\ddot{x} + h\dot{x}^2 + x = 0$

Let

$$\frac{dx}{dt} = y$$

then

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = M\frac{dx}{dt}$$

Thus

$$\ddot{x} + h\dot{x}^2 + x$$

$$= M\dot{x} + h\dot{x}^2 + x = 0$$

or $$h\dot{x}^2 + M\dot{x} + x = 0$$

Both roots of this equation are valid. On the other hand, if one substitutes $\dot{x} = Kx$ this equation becomes

$$h(Kx)^2 + MKx + x = 0$$

or

$$M = \frac{-1-K^2hx}{K}$$

which is simpler to plot for any given values of h and K. Since the phase trajectories are found to be symmetrical about the x-axis, only the upper half of the portrait is shown in Fig. 2-26 for the case h = 1.

$$\text{for} \quad h = 1 \quad \begin{cases} M = \dfrac{-1-K^2x}{K} \\[2ex] \dot{x} = Kx \end{cases}$$

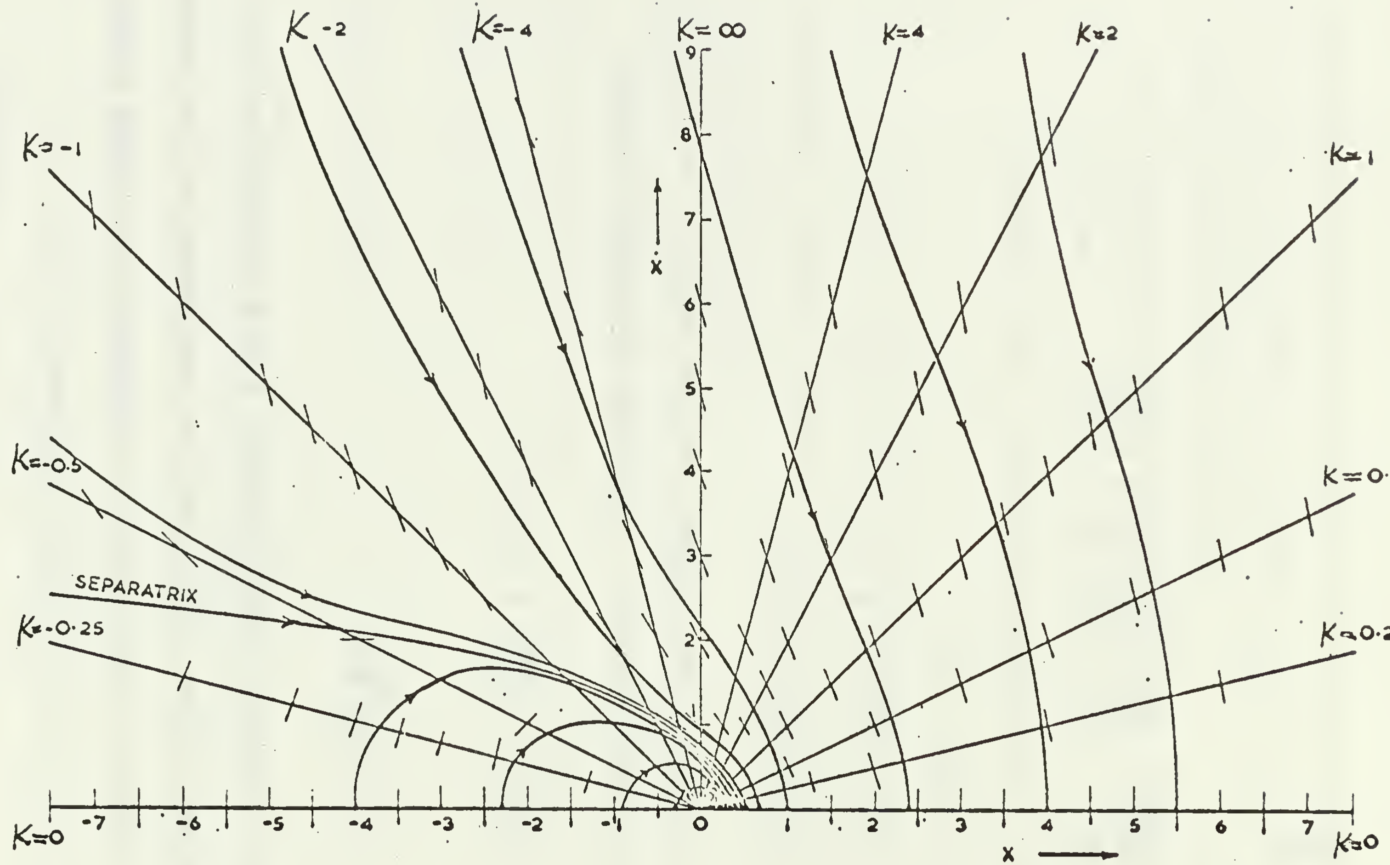


Fig. 2-26. Phase trajectories for $\ddot{x} + \dot{x}^2 + x = 0$.

$$\text{at } K = 1 \begin{cases} \dot{x} = x \\ M = -1-x \end{cases}$$

$$K = 2 \begin{cases} \dot{x} = 2x \\ M = 2x - 0.5 \end{cases}$$

$\vdots$

etc.

(Example 2) Let it be required to find the behavior of the following time-varying system:

$$t^2\ddot{x} + t\dot{x} + x^3 = 0$$

for all pertinent initial conditions (with $t \neq 0$).

The task would be enormous if it were to be attacked by the phase-plane method. Using the $t\dot{x}$-x plane the differential equation reduces to

$$t\dot{x} = y = -\frac{x^3}{M}$$

Note:

Let

$$t\dot{x} = y$$

then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = M\dot{x} = \dot{x} + t\ddot{x}$$

$$Mt\dot{x} = t\dot{x} + t^2\ddot{x} = -x^3$$

or

$$t\dot{x} = y = -\frac{x^3}{M}$$

Taking different values for M the isoclines (enlarged or reduced scale plot of the cubic curve) are drawn and the respective slopes marked on them with their direction (see Fig. 2-27).

3. Discussion

a. Simple substitutions like $\dot{x} = Kx, Kx^2$, $\dot{x}^2 + x^2 = \gamma^2$ in the isocline equations of second-order systems simplify plotting of the phase trajectories to a great extent. The method is a useful supplement of the isocline method.

b. Since there are not many methods to evaluate quickly the bahavior of second-order non-linear time-varying systems for various pertinent initial conditions and/or for different non-linear characteristics, Deekshatulu suggests, in general time-varying coordinate planes, the use of $t\dot{x}$-x and $t\dot{x}^2$-x planes for obtaining easily the solution of certain classes of nonlinear time-varying systems.

H. PELL'S METHOD

1. Method-Description

Quite generally, second-order nonlinear autonomous equations of the form

$$\frac{d^2x}{dt^2} + g(x, \frac{dx}{dt}) = 0$$

may be written as

$$\frac{d^2x}{dt^2} + \phi(\frac{dx}{dt}) + f(x) = 0. \tag{2-31}$$

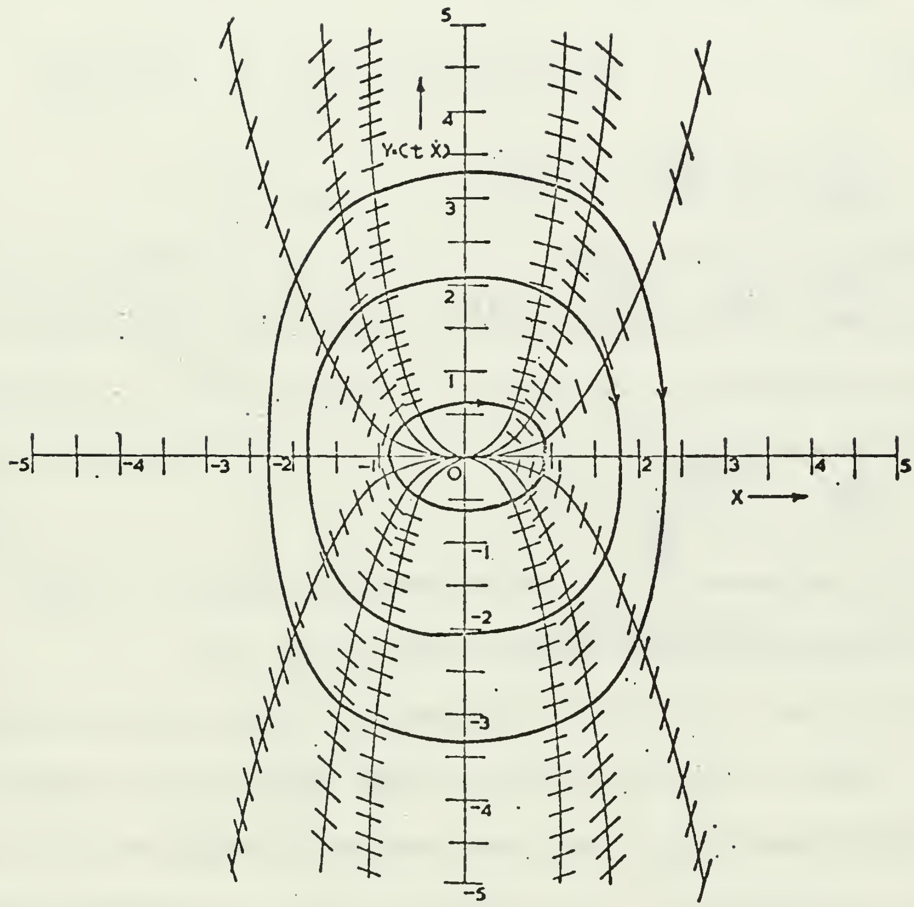


Fig. 2-27. Trajectories for $t^2\ddot{x} + t\dot{x} + x^3 = 0$.

Let

$$y = \frac{dx}{dt}$$

then

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Thus

$$\frac{d^2x}{dt^2} + \phi\left(\frac{dx}{dt}\right) + f(x)$$

$$= \frac{dy}{dx} \cdot \frac{dx}{dt} + \phi\left(\frac{dx}{dt}\right) + f(x) = 0$$

or

$$\frac{dy}{dx} = \frac{-\phi\left(\frac{dx}{dt}\right) - f(x)}{\frac{dx}{dt}}$$

$$= \frac{-\phi(y) - f(x)}{y} \qquad (2\text{-}32)$$

Now, as shown in Fig. 2-28, plot $-\phi(y)$ versus y and $-f(x)$ versus x. Given the initial conditions $x(0)$ and $y(0)$, the construction of a segment of the trajectory through the initial condition proceeds as follows:

a. Move vertically from IC and find $f[x(0)]$; lay this off on the x-axis from $x(0)$ by use of a compass or a 45° triangle. This establishes point A.

b. Draw line AB, where $OB = y(0)$.

c. Find $-\phi[x(0)]$ by moving horizontally and draw CD parallel to AB.

d. Now the hypotenuse (D) (IC) of a right triangle whose base is the distance DE = $\phi[y(0)] + f[x(0)]$ and whose altitude is the distance (E) (IC) = $y(0)$ is established. The slope of the hypotenuse is

$$\frac{(E)\ (IC)}{DE} = \frac{y(0)}{\phi[y(0)] + f[x(0)]} \tag{2-33}$$

e. The negative reciprocal of Eq. (2-33) is the slope required in Eq. (2-22); thus a line perpendicular to the hypotenuse, passing through the initial condition forms a segment of the required trajectory. The usual precautions must be observed as to the length of the extrapolated segment.

Note: The major advantages of this method are:

(1) Its simplicity. Only two triangles are needed to carry out the solution once the two functions are plotted.

(2) It eliminates the trial and error of the delta method.

2. <u>Example</u>

Consider the equation (the hydraulic speed control system)

$$\frac{d^2e}{dt^2} + c_1 \frac{de}{dt} + \omega_o^2(1+a^2e^2)e + \frac{Ke}{c_2} = 0 \tag{A}$$

where

c_1 = viscous damping coefficient

c_2 = inertial coefficient related to volume of nozzle chambers

K = gain of amplifier

ω_o = natural frequency of vibration of flapper for small amplitudes.

The equivalent flapper spring is assumed to be linear. Eq.(A) may be rewritten as

$$\frac{d^2e}{dt^2} + c_1 \frac{de}{dt} + \omega_1^2(1+b^2e^2)e = 0 \tag{B}$$

Let $\tau = \omega_1 t$ to change time scale; thus Eq.(B) becomes

$$\frac{d^2e}{d\tau^2} + \frac{c_1}{\omega_1}\frac{de}{d\tau} + (1+b^2e^2)e = 0 \tag{C}$$

Choose the following more or less arbitrary set of coefficients designed to emphasize the construction technique:

$c_1 = 164$ $\qquad\qquad$ $\omega_o = 377$

$c_2 = 0.01$ $\qquad\qquad$ $a^2 = 0.858$

$K = 1000$

on a consistent set of units. Then

$$\omega_1^2 = \omega_o^2 + \frac{K}{c_2} = 242000$$

$$b^2 = \frac{\omega_o^2}{\omega_1^2} = 0.58$$

$$a^2 = 0.5$$

$$\frac{c_1}{\omega_1} = 0.33$$

Let

$$x = e$$

$$y = \frac{de}{d\tau}$$

Eq.(C) becomes

$$\frac{dy}{dx} = \frac{-0.33y - (1 + 0.58x^2)x}{y}$$

$\phi(y)$ and $f(x)$ may be identified and plotted and the construction completed as shown in Fig. 2-29.

3. Discussion

This method is satisfactory:

a. If only a single solution curve is needed.

b. Only a few of these line segments are actually put to use.

I. SOMAYAJULU'S METHOD

In this method a graphical procedure is suggested for drawing phase-plane trajectories making use of tangent slopes instead of normals (i.e., the negative reciprocal of tangent slope values) as is done now in the Liénard construction.

1. Method-Description

A straightforward construction is presented for more general cases to which the isocline method is not applicable and construction of individual trajectories is required.

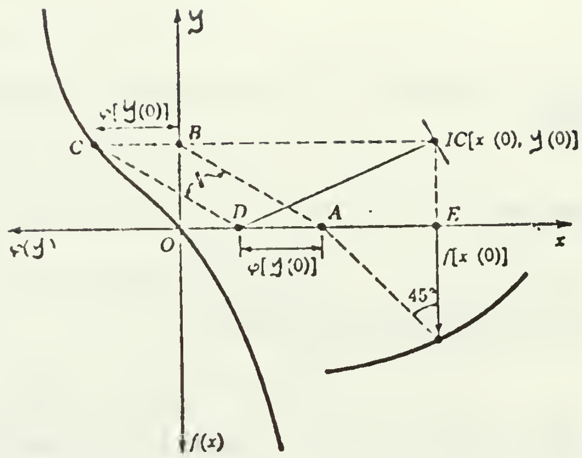


Fig. 2-28. Pell's Method of phase-trajectory construction.

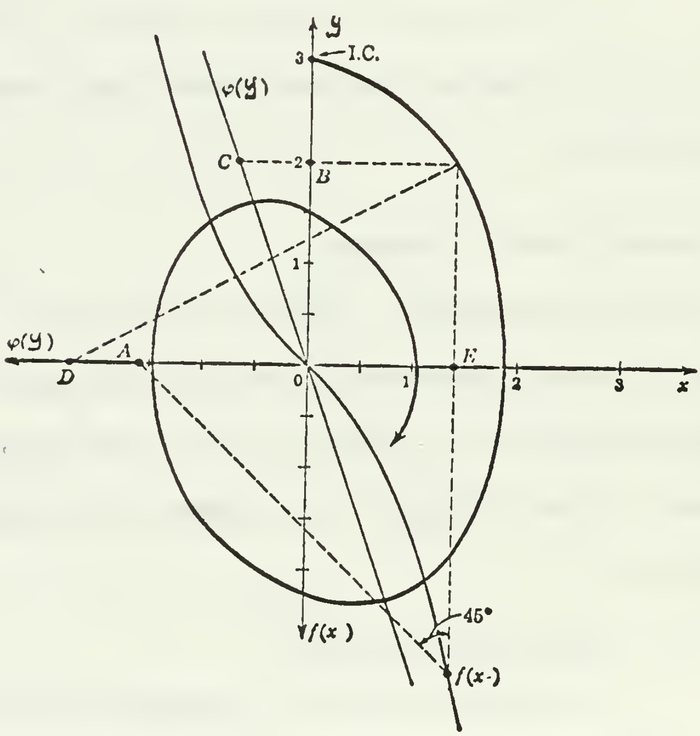


Fig. 2-29. Example construction using Pell's method for the hard-spring system.

a. Let the slope of a second-order system be given by

$$M = \frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}}$$

$$= \frac{f_1(x) + f_2(\dot{x})}{g_1(x) + g_2(\dot{x})} \qquad (2\text{-}34)$$

Assume f_1, f_2, g_1, and g_2 are all continuous, single-valued and of the odd characteristic type; these curves are first drawn on the phase-plane.

Referring to Fig. 2-30, the initial point $P(x_o,\dot{x}_o)$ is marked. Make $PQ = f_1(x_o)$, $QR = f_2(\dot{x}_o)$. Similarly the length RST is made equal to $g_1(x_o) + g_2(\dot{x}_o)$. The angle PRT is a right angle. Note that the horizontal line RST is drawn to the left of PR since the slope is known to be positive from the nature of the functions f_1, f_2, g_1, and g_2 and the initial conditions x_o and $\dot{x}_o$. A small length of the straight line joining P and T gives the required slope at P. The field direction is marked on the basis that with $\dot{x}$ positive x should increase. It is seen in the procedure that it is necessary to first determine the direction of the normal to the phase trajectory.

b. Consider equation

$$\ddot{x} + f(x)\, h(\dot{x}) + g(x) = 0 \qquad (2\text{-}35)$$

This equation is considered to be of practical importance because of the presence of nonlinear damping and nonlinear restoring forces.

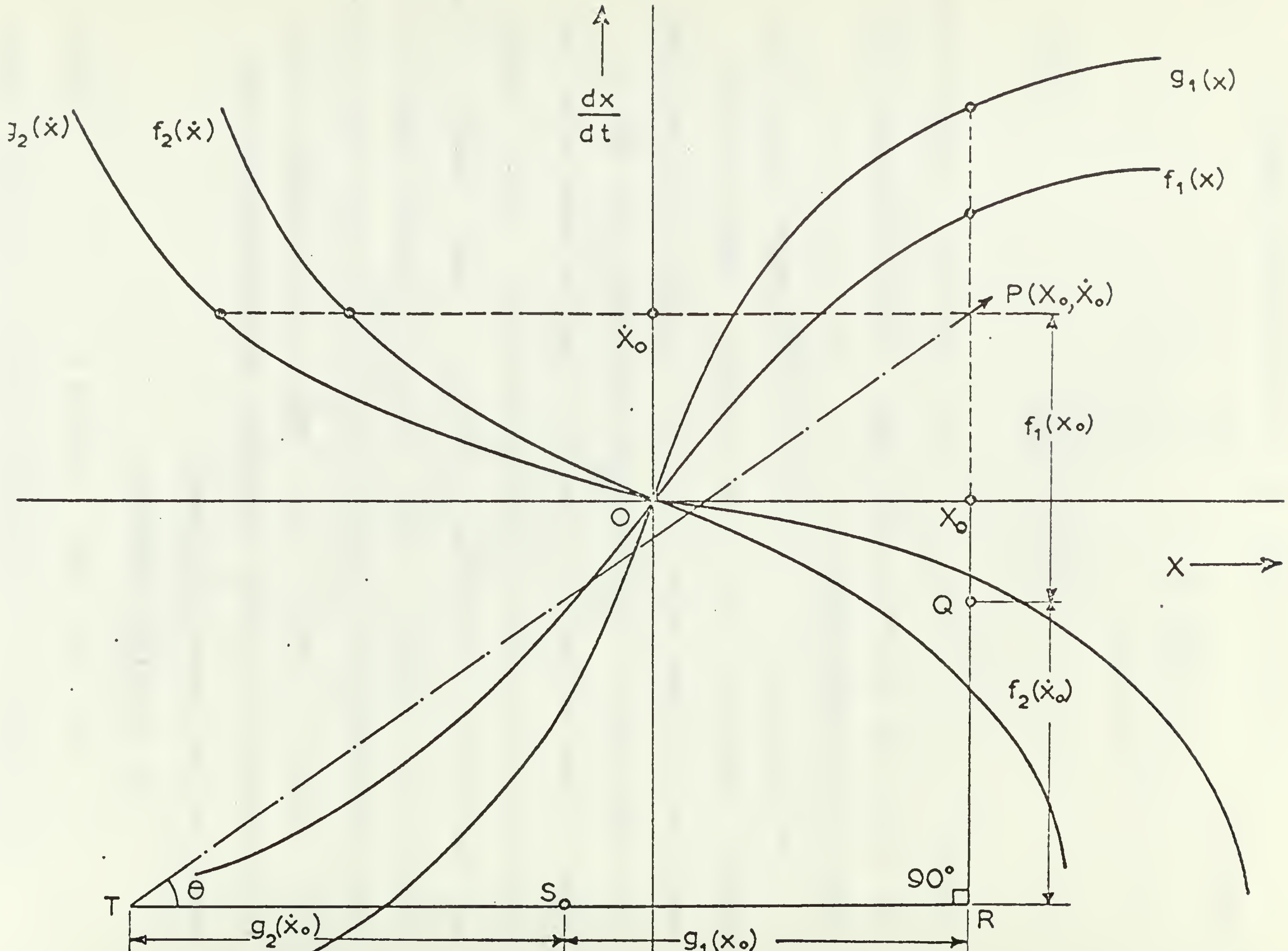


Fig. 2-30. Graphical determination of field direction.

Let

$$y = \frac{dx}{dt} \quad \text{and}$$

rewrite the equation (2-35) in form (2-36)

$$M = -\frac{f(x)\ h(\dot{x}) + g(x)}{\dot{x}} \tag{2-36}$$

The plots of $f(x)$, $h(\dot{x})$ and $g(x)$ are shown in Fig. 2-31. Also a parabola $\dot{x}^2$ or x^2 (only one half will suffice) is drawn in addition to the above curves.

Let the initial point be at $P(x_o, \dot{x}_o)$;

then:

$OS = x_o \quad , \quad OQ = \dot{x}_o$

$QR = h(\dot{x}_o) \quad , \quad SB = f(x_o)$

and $SA = g(x_o)$

Make $SM = QR$.

Draw a circle with BM as a diameter and let it intersect the x-axis at the points C, C.

Then

$$SC = \sqrt{f(x_o)\ h(\dot{x}_o)}$$

With the help of the parabola the squared length of SC is obtained and is marked as PN.

Make $\quad NL = g(x_o)$

and $\quad LK = \dot{x}_o$

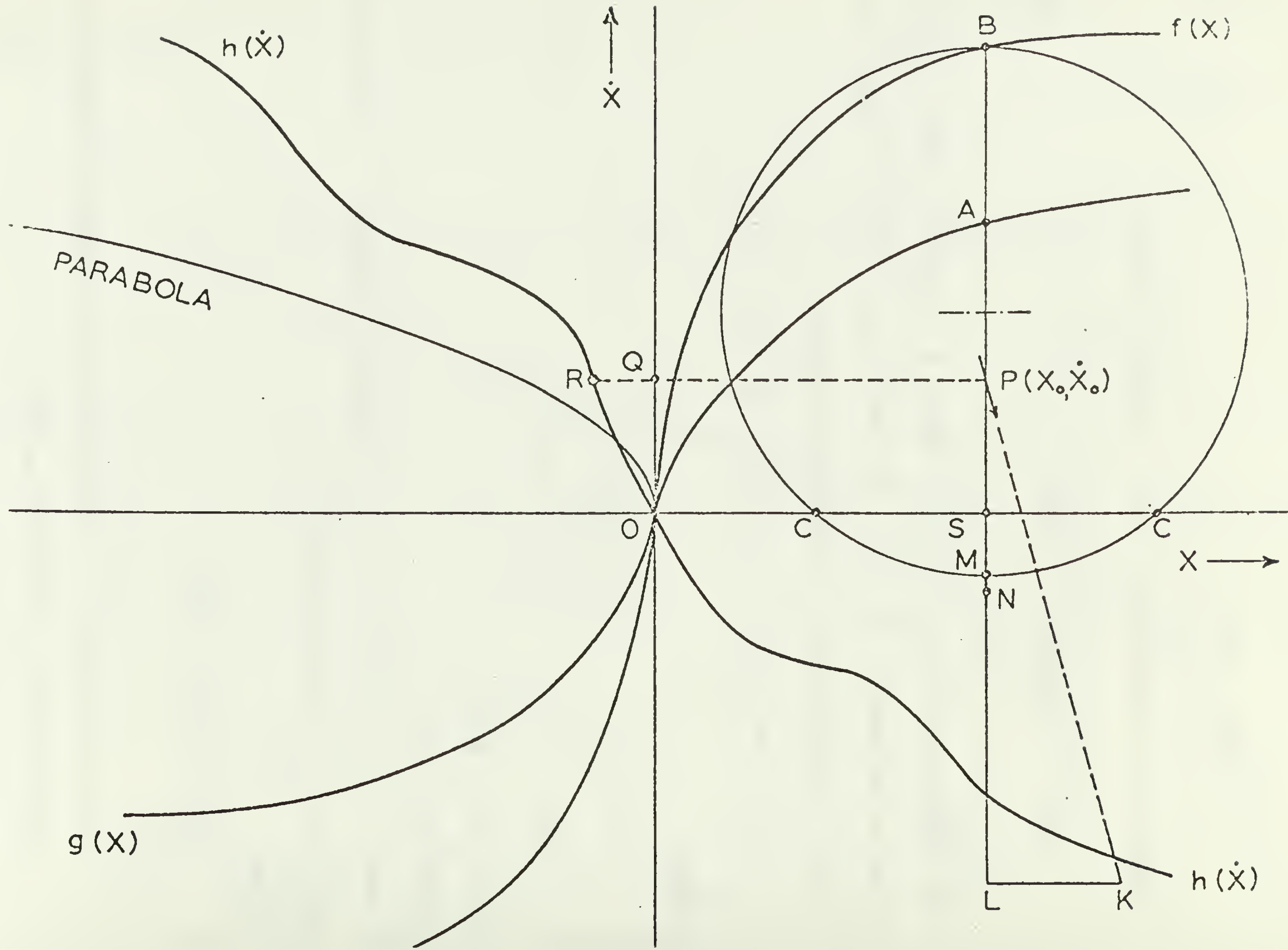


Fig. 2-31. Construction details.

The segment line through P joining P and K represents the required tangent slope of the phase trajectory at P. The direction of motion is decided to be downwards. The new values x_1 and $\dot{x}_1$ at the end of this segment are determined and the procedure repeated.

Note: (a) In both the above mentioned methods it is necessary to take equal scales on the x and $\dot{x}$ axes.

(b) The accuracy in plotting is enhanced by considering small segments.

2. Example

$$\ddot{x} + \dot{x}^2 + x = 0$$

with initial condition $x_o = -4$ and $\dot{x}_o = 0$.

Rewrite $\ddot{x} + \dot{x}^2 + x = 0$ to the form

$$M = \frac{-\dot{x}^2 - x}{\dot{x}} = \frac{\dot{x}^2 + x}{-\dot{x}}$$

and draw

$$f_1 = \dot{x}^2 \qquad f_2 = x$$

$$g_1 = -\dot{x}$$

on the phase-plane (see Fig. 2-32) (using Somayajulu's method as in I-1-a).

3. Discussion

This method is based on geometrical relationships, so a precise step-by-step procedure must be followed.

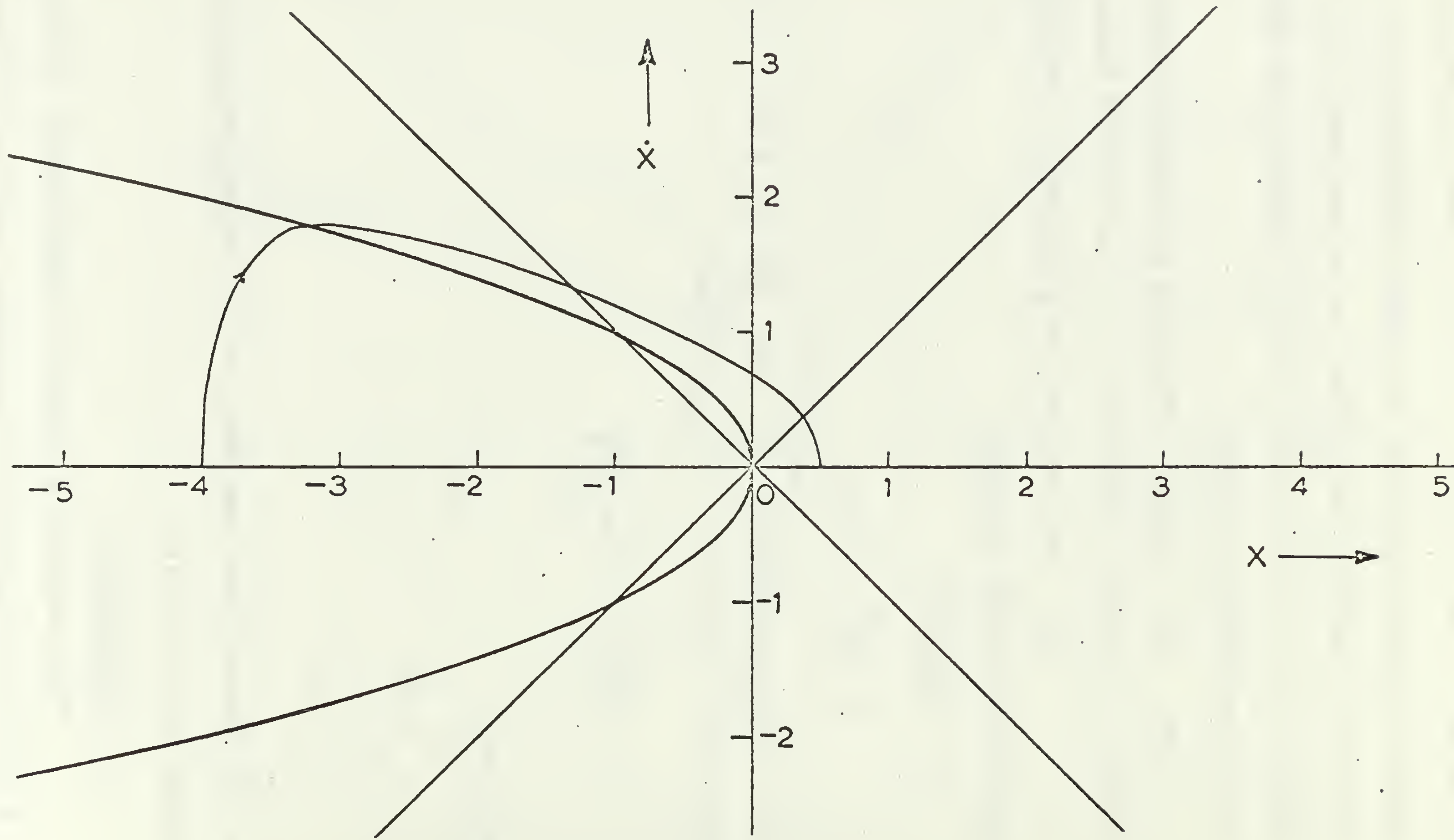


Fig. 2-32. Example $\ddot{x} + \dot{x}^2 + x - 0.$

J. THE E-FUNCTION METHOD

1. Method-Description

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x\right) + g(x) = h(t) \qquad (2\text{-}45)$$

Let the Kinetic energy and potential energy be defined as $(\frac{1}{2})\dot{x}^2$ and $Q(x)$

where

$$Q(x) = \int_0^x g(x)\,dx. \qquad (2\text{-}46)$$

Define the total energy as

$$E = \frac{1}{2}(\dot{x})^2 + Q(x). \qquad (2\text{-}47)$$

Differentiating E with respect to time

$$\frac{DE}{dt} = \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + \frac{dQ(x)}{dt}$$

$$\dot{E} = \dot{x} \cdot \ddot{x} + \dot{Q}(x) \qquad (2\text{-}48)$$

Differentiating E with respect to x

$$\frac{dE}{dx} = \frac{d}{dx}\left[\frac{1}{2}(\dot{x})^2\right] + \frac{dQ(x)}{dx}$$

$$= \frac{dx}{dt}\left[\frac{d}{dx}\left(\frac{dx}{dt}\right)\right] + \frac{dQ(x)}{dx} \qquad (2\text{-}49)$$

$$= \ddot{x} + g(x)$$

From Eq. (2-49)

$$\frac{d^2x}{dt^2} = \frac{dE}{dx} - g(x)$$

Substituting this into Eq. (2-45) yields

$$\frac{dE}{dx} - g(x) + f(\frac{dx}{dt}, x) + g(x) = h(t),$$

or

$$\frac{dE}{dx} = h(t) - f(\dot{x}, x) \qquad (2\text{-}50)$$

From Eq. (2-47), an E-x plot can be made for different values of $\dot{x}$. These curves may be called equi-velocity curves, as $\dot{x} = v$, and they represent different kinetic energy levels. For $\dot{x} = 0$, the E-x curve is simply $E = Q(x)$ where $Q(x)$ represents the potential energy. At $\dot{x} = x_1$, the difference between any two E-x curves, say, between the curves for $v = v_1$ and $v = v_o$ is equal to the difference $(v_1^2 - v_o^2)/2$.

The value of $v = \dot{x}$ is, from Eq. (2-47)

$$\dot{x} = v = \pm\sqrt{2[E-Q(x)]}. \qquad (2\text{-}51)$$

Substituting Eq. (2-51) into (2-50),

$$\begin{aligned} \frac{dE}{dx} &= h(t) - f\left[x, +\sqrt{2[E-Q(x)]}\right] \quad \text{for} \quad v > 0 \\ \frac{dE}{dx} &= h(t) - f\left[x, -\sqrt{2[E-Q(x)]}\right] \quad \text{for} \quad v < 0 \end{aligned} \qquad (2\text{-}52)$$

The E-function method is thus a method for constructing the E-x trajectory according to Eq. (2-50). See Fig. 2-33, the slope of AB is equal to $h(t)-f(x,v_{av})$, where

$$v_{av} = \frac{v_1 + v_2}{2}$$

and x should be corrected to $\frac{x_1 + x_2}{2}$, even though it is initially taken as equal to x_1.

2. <u>Example: Van der Pol Equation</u>

$$\ddot{x} - (1 - x^2)\dot{x} + x = 0$$

E-x plot (see Fig. 2-34).

From E-x plot get the $v(v = \dot{x})$

Note: As compared with the isoclines and trajectories of the Van der Pol equation obtained from the digital computer, errors exist in this method. (See Fig. 2-35)

K. THE SLOPE-LINE METHOD (See Fig. 2-36)

Given an equation

$$\frac{dy}{dx} = f(x)$$

or $dy = f(x)dx$, the incremental relations may be written

$$\Delta y = f_{av}\Delta x \tag{2-53}$$

where $f_{av} = \frac{f_1 + f_2}{2}$ represents the average value of f(x) during the interval.

Thus

$$\begin{aligned} \Delta y &= \frac{\Delta x}{2} f_1 + \frac{\Delta x}{2} f_2 \\ &= \Delta y_1 + \Delta y_2 \end{aligned} \tag{2-54}$$

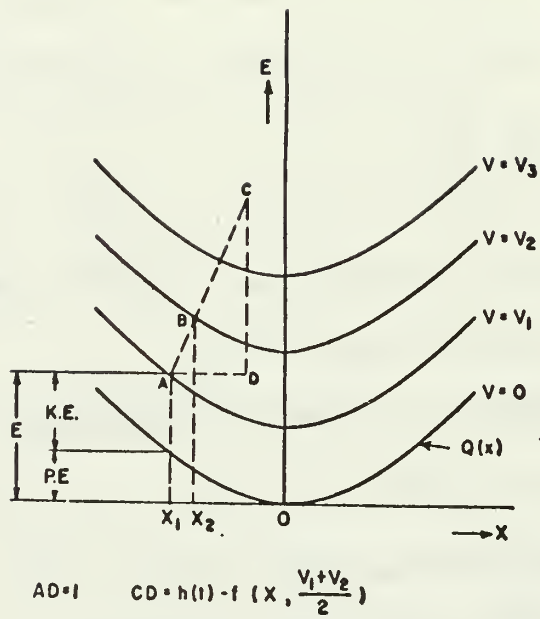


Fig. 2-33. The E-function method.

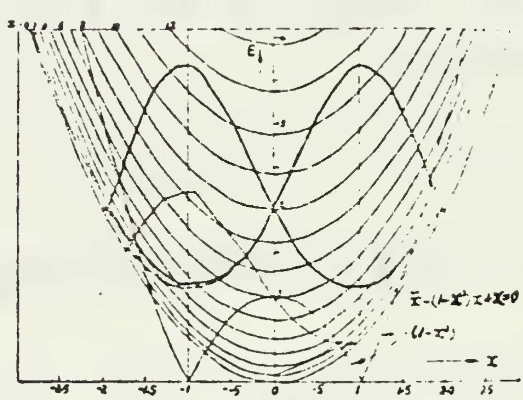


Fig. 2-34. Example (II-J-2).

Let an angle θ be chosen such that $\tan\theta = \frac{\Delta x}{2}$, for a chosen value of Δx, then

$$\tan\theta = \frac{\Delta y_1}{f_1}$$

$$= \frac{\Delta y_2}{f_2} \qquad (2\text{-}55)$$

Starting from an ordinate f_1, make an angle θ and draw a dotted line bc to intersect the horizontal line at c. Then ac = Δy. From c, draw another dotted line, making the same angle θ with the vertical line erected from c, until another ordinate f_2 is intersected. As shown de = f_2 can be projected from the f(x) curve on the left of the figure.

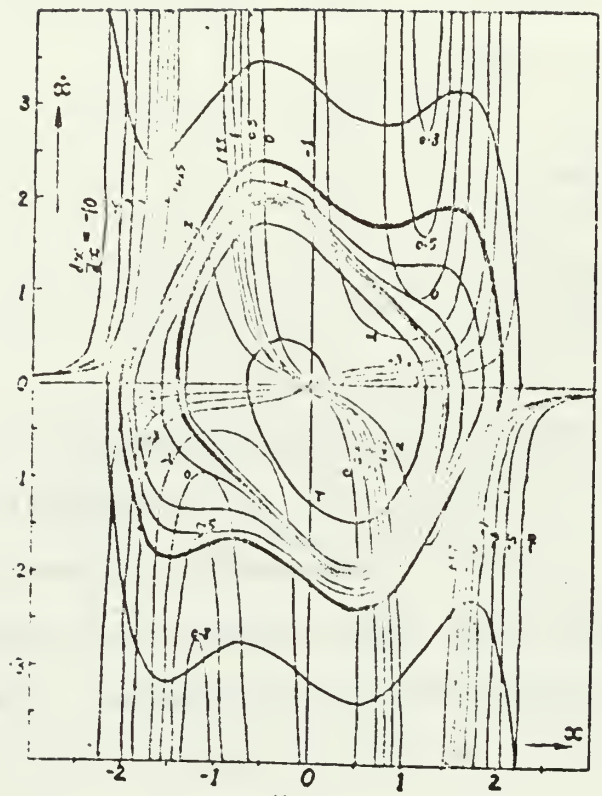
Then ce = Δy_2, and ae = Δy, corresponding to the increment Δx shown on the left of the figure.

This method can be applied to the solution of simultaneous first-order differential equations, linear or nonlinear. For a first-order nonlinear equation

$$\frac{dy}{dx} + g(x) = h(x) \qquad (2\text{-}56)$$

the incremental relations give

$$\Delta y = [h(x) - g(y)_{av}]\Delta x \qquad (2\text{-}57)$$



isoclines

and

trajectories

Fig. 2-35.(Example II-J-2)

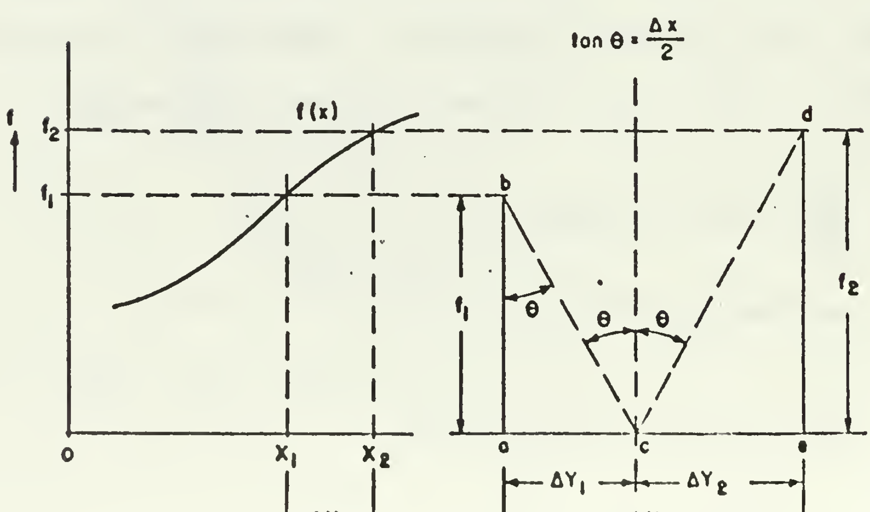


Fig. 2-36. The slope-line method.

For example, in a problem of surge tank transients, the equations are of the following form:

$$- \frac{du}{dT} = y + f(u) \qquad \text{(A)}$$

$$- \frac{dy}{dT} = v(T, y) - u \qquad \text{(B)}$$

When the equations are in normalized form both the horizontal and vertical systems of slope lines can be drawn with the same slope. Eq.(A) represents acceleration. Eq.(B) is obtained from the condition of continuity.

L. COMPARISON

It is readily seen that some methods are easier than others but may be of limited usefulness, applying only to some particular problems. Some of them are useful but complicated, so that it is easy to make a mistake. It seems the most useful and general method is the isocline method, when one solves problems by hand labor. Deekshatulu's and Murthy's methods are useful supplements. The difficulty of the isocline method lies in the labor required to get the isoclines. So if we can use the digital computer to generate a program, call upon it to draw the isoclines and phase trajectories on the same phase-plane, then the isocline method is more useful and general than other methods.

For time-varying systems, the Delta method and Pell's method are useful. Deeskshatulu's transformation method (by using a new plane) is a useful supplement.

III. THE APPLICATION OF PHASE-PLANE METHODS

Phase-plane analysis of nonlinear system problems not only gives a convenient display for interpreting results, but also permits use of different techniques for shaping the phase trajectory to make it satisfy design requirements. So, it can be said that the phase plane has become a logical tool in the solution of problems.

Generally, phase-plane analysis is limited to second-order systems. Since higher-order derivatives cannot be plotted on the phase plane, higher-order systems would not be completely defined. Of course, the method might be extended from phase plane to phase spaces of n dimensions to handle nth-order systems, but then it loses some of its simplicity and convenience.

In this chapter, the manipulations of the phase plane are discussed, then the uses of different graphical methods and techniques for phase-plane analysis of nonlinear system problems. All the problems are of second order or can be approximated by second-order differential equations.

A. MANIPULATIONS ON THE PHASE PLANE

1. Elementary Algebraic Manipulations

a. Addition

Addition of phase-plane trajectories is accomplished in a fairly straightforward manner. Suppose we have two quantities x_1 and x_2, and it is desired to find the phase-plane trajectory of $x_1 + x_2$.

Let x and $\dot{x}$ represent phase-plane coordinates: therefore

$$x_t = x_1 + x_2$$

$$\dot{x}_t = \dot{x}_1 + \dot{x}_2 \tag{4-1}$$

On the graph, simply add the x's to obtain the new x at a particular time and the $\dot{x}$'s to obtain the new $\dot{x}$. The time for each plot must be known at each point so that addition takes place at points of isochronism.

b. Multiplication:

If the trajectories are x_1 and x_2 then the products are:

$$x_t = x_1 \cdot x_2 \tag{4-3}$$

$$\dot{x}_t = x_1\dot{x}_2 + x_2\dot{x}_1 \tag{4-4}$$

Again the addition and multiplication must take place at points of isochronism.

c. Raising to a power:

For nth power

$$x_t = x^n \tag{4-5}$$

$$\dot{x}_t = nx^{n-1}\dot{x} \tag{4-6}$$

The plot is taken point by point and the x coordinate is raised to the nth power to obtain the new x_t; the x coordinate is raised to the x^{n-1} power and multiplied by $n\dot{x}$ in order to obtain the new $\dot{x}_t$.

d. Operation on the trajectory with a function:

If the trajectory for x is known, then the trajectory for f(x) can be founded as follows:

$$x_t = f(x) \tag{4-7}$$

$$\begin{aligned} \dot{x}_t &= \frac{df(x)}{dx} \; \frac{dx}{dt} \\ &= \frac{df(x)}{dx} \cdot \dot{x} \end{aligned} \tag{4-8}$$

The new x is the value of f(x) and the new $\dot{x}$ is the slope of the f(x)-vs-x curve times $\dot{x}$.

2. Differentiation

The trajectory x is known and the trajectory of the derivative of x is desired. This has a practical application. Suppose a device or system output is to be displayed as a phase-plane trajectory. This can be done by differentiating the output x, thus obtaining $\frac{dx}{dt}$ or $\dot{x}$. Then $\dot{x}$ is plotted against x. Unfortunately, it is very difficult to differentiate properly; and alternate is to take the integral of the trajectory, then differentiate the trajectory graphically to obtain the true $\dot{x}$-x plot.

For illustration purposes, let

$$u = \int x \, dt \; ; \tag{4-9}$$

where

$$x = \frac{du}{dt} \; .$$

Then

$$\dot{x} = \frac{dx}{dt}$$

But

$$\frac{dx}{dt} = \frac{dx}{du} \; \frac{du}{dt} = \frac{dx}{du} \cdot x$$

So in order to find the new $\dot{x}$ for each point x and u:

a. Find the slope $\frac{dx}{du}$.

b. Multiply this value by the value of x at that point.

c. Plot the new $\dot{x}$ on the -u axis. (See Fig. 3-1)

3. Transformation

It is possible to transform the phase-plane trajectory by applying some transformation to $\dot{x}$ and x quantities for some purpose.

For example:

$$\dot{x}_2 = f(\dot{x}_1, x_1)$$
$$x_2 = g(\dot{x}_1, x_1)$$

An example of this might be

$$\dot{x}_2 = \dot{x}_1 \sin x_1$$

$$x_2 = \dot{x}_1 \cos x_1$$

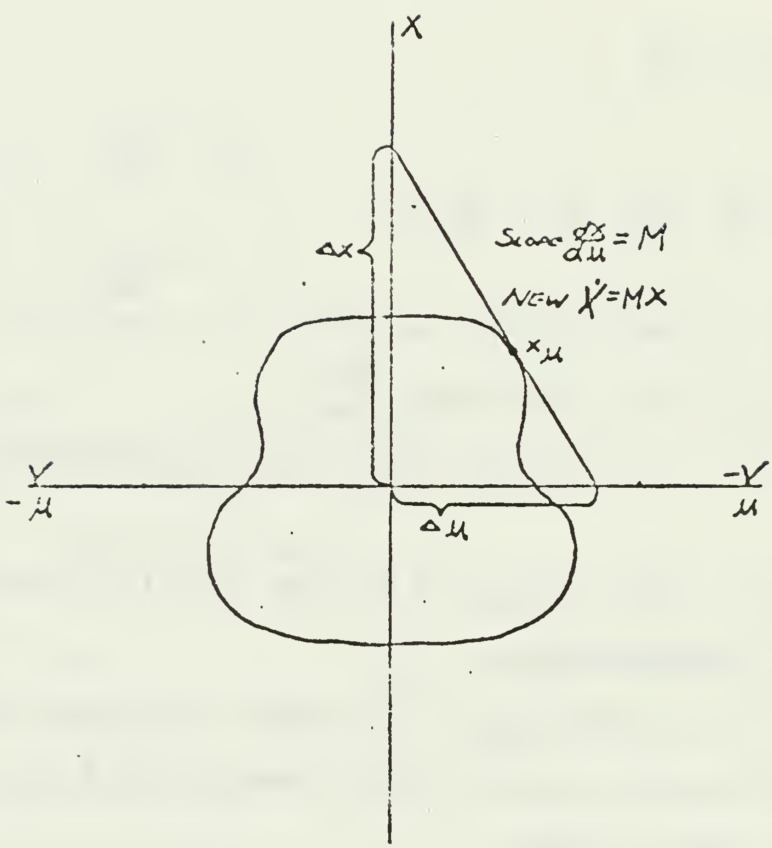


Fig. 3-1. Differentiation procedure.

4. <u>Methods of Shaping Phase-Plane Response</u>

There are several ways in which a phase-plane response can be shaped to conform to design requirements.

These are:

(1) Choosing specific nonlinearities.

(2) Limiting.

(3) Piecewise shaping.

a. Choosing Specific Nonlinearities:

For the following second-order differential equation:

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x, \dot{x}) = 0$$

various combinations $f(x,\dot{x})$ and $g(x,\dot{x})$, may be chosen to obtain a required response.

b. Limiting:

The values of x and $\dot{x}$ may be restricted to certain values as shown in Fig. 3-2. The limiting may be extended to $\dot{x}$ also or even to some function of x and $\dot{x}$.

c. Piecewise Shaping:

See Fig. 3-3.

B. COMMON PHYSICAL NONLINEARITIES

1. <u>Saturation: (the most common of all nonlinear phenomena</u>)

Consider the case of a saturating electronic amplifier driving a motor in an instrument servo. (See Fig. 3-4(a) and (b))

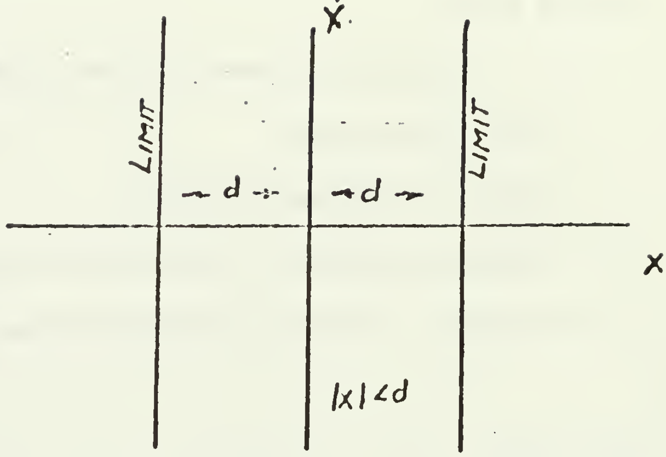


Fig. 3-2. Limiting of x on the phase-plane.

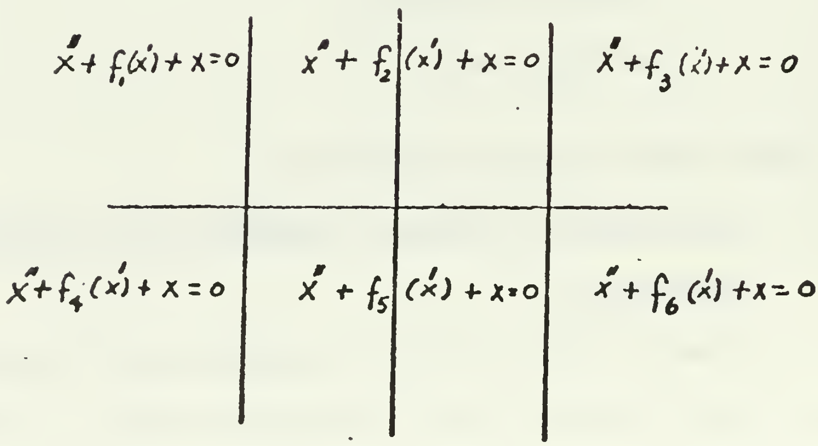
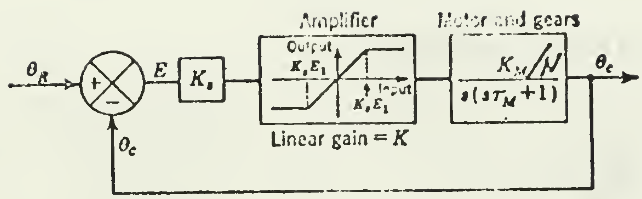
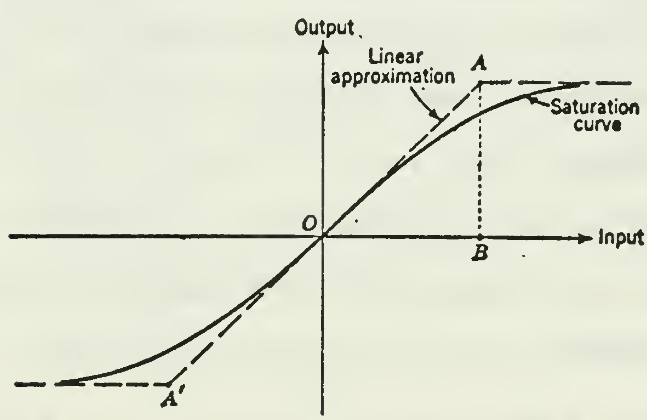


Fig. 3-3. Piecewise shaping of trajectory.



(a) Block diagram of a servo
with saturation.



(b) Input vs output characteristics.

Fig. 3-4. Common physical nonlinearities - Saturation.

The equation in the linear region is

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} + \frac{K_s K K_m}{N\tau_m} E = 0$$

$$-E_1 < E < + E_1$$

For saturation operation:

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} = + c \qquad E \leq - E_1$$

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} = - c \qquad E \geq + E_1$$

where

$$c = \frac{KK_s K_m}{N\tau_m} E_1 .$$

Using the isocline method, draw isoclines on the phase-plane $\dot{E}$ vs E. In the linear region the isoclines are radial straight lines (see Fig.3-4c), emanating from a focus at the origin.

While in the saturation region the isoclines are horizontal, the transition from linear to nonlinear operation is indicated by a vertical line at $E = E_1$. Dividing lines on the phase plane may be obtained from many nonlinear phenomena.

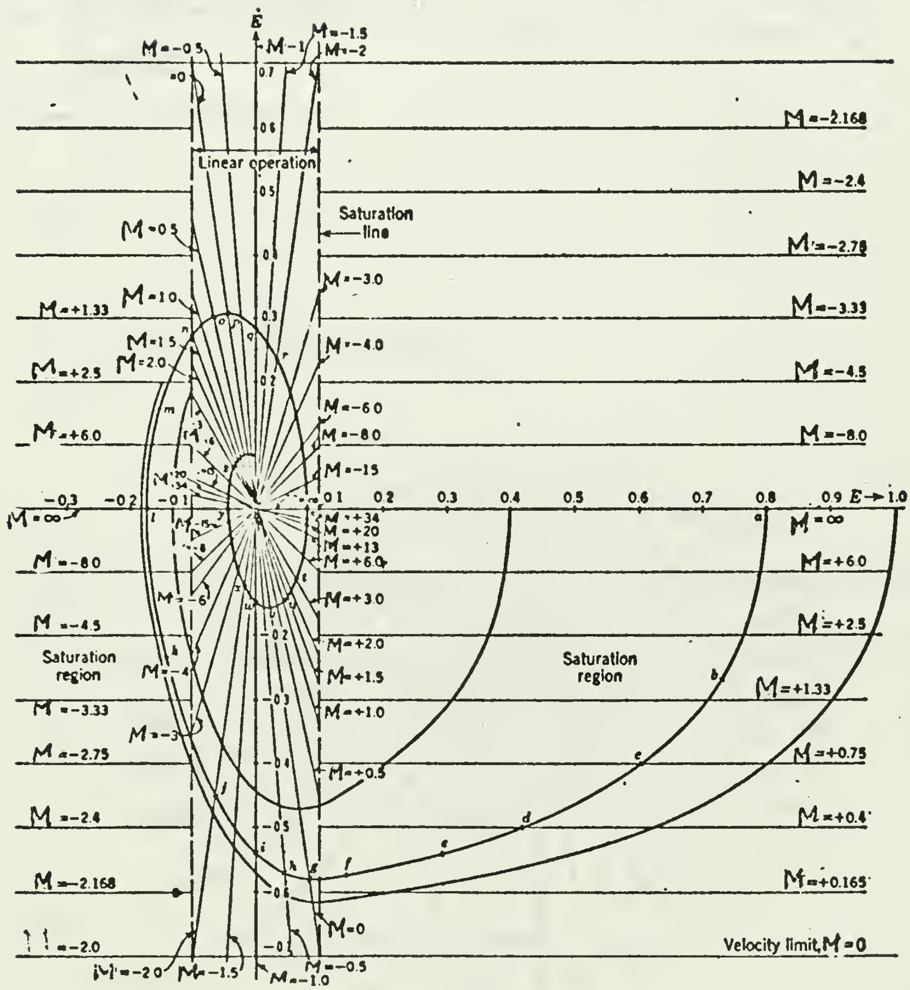
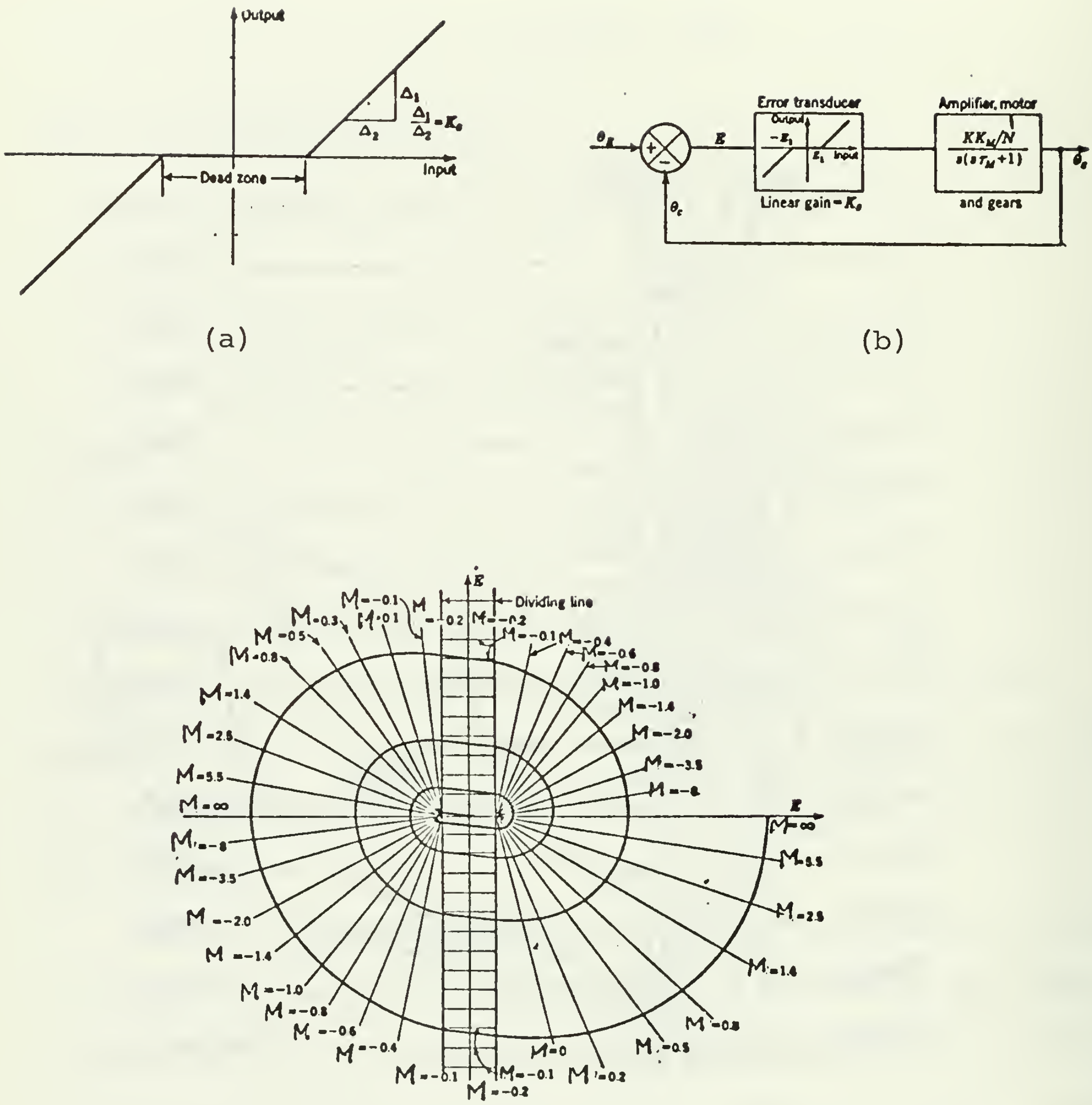


Fig. 3-4.

(c) Isoclines, dividing lines and trajectory.



(c)

Fig. 3-5. Dead zone - (a) Input vs output characteristics (b) Block diagram of a servo with dead zone (c) Isoclines, dividing lines and trajectory.

2. <u>Dead Zone</u>

This arises in mechanisms which are spring-loaded to minimize backlash and in many other devices which are insensitive to small signals. Consider a servo with a dead zone in the error transducer.

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} + E = 0$$

$$-E_1 < E < +E_1$$

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} + \frac{K_s K K_m}{N\tau_m}(E - E_1) = 0$$

$$|E| > E_1$$

In the dead zone, there is no control forcing-function, so in this case isoclines are straight lines with slope

$$M = \frac{-E - \dot{E}\frac{1}{\tau_m}}{\dot{E}}$$

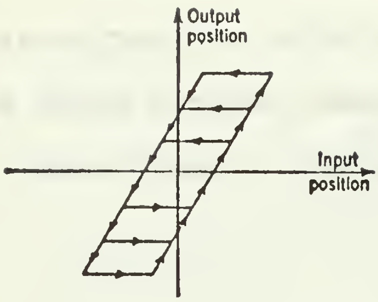
(See Fig. 3-5).

3. <u>Backlash</u>

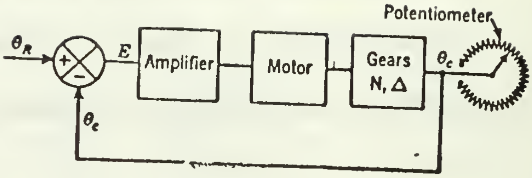
Most mechanical linkages exhibit a certain amount of backlash. (See Fig. 3-6)

Consider a simple positioning servo with backlash. When the backlash is taken up, the operation is linear. The equation of the system is

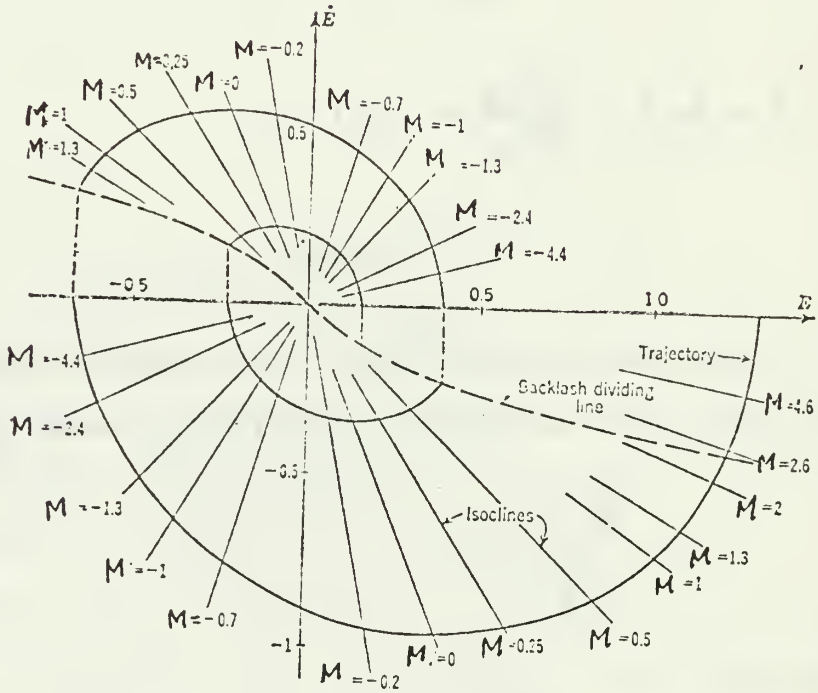
$$\ddot{E} + 2\zeta\omega_n\dot{E} + \omega_n^2 E = 0$$



(a)



(b)



(c)

Fig.3-6. Common physical nonlinearities - Backlash.

When zero velocity is reached and the motor reverses, the output shaft remains stationary until the backlash is taken up. Thus the error remains fixed at some value E_1. The equation of this motion is

$$\ddot{\theta}_m + 2\zeta\omega_n\dot{\theta}_m = E_1$$

4. <u>Instrument Servos with Coulomb Friction and Stiction</u>

a. A second-order servo with coulomb friction, (see Fig. 3-7(a)).

The error equation for a step-displacement input is

$$J\ddot{E} + f\dot{E} + KE + C\,\mathrm{Sgn}(\dot{E}) = 0$$

from which the isocline equation becomes

$$\dot{E} = -\frac{KE}{MJ+f} - \frac{C\,\mathrm{Sgn}(\dot{E})}{MJ+f}$$

Note: Let $\dot{E} = y$

$$\ddot{E} = \frac{dy}{dt} = \frac{dy}{dE}\,\frac{dE}{dt} = M\dot{E}$$

The second term introduces the effect of the nonlinearity. This effect is seen to be translation of the focal point to $E = +\frac{C}{K}$ when $\dot{E}$ is negative and to $E = -\frac{C}{K}$ when $\dot{E}$ is positive. $\dot{E} = 0$ (or E axis) is a dividing line. (See

Fig. 3-7). The trajectory terminates at D because the drive torque produced by the error at D is less than the coulomb torque C.

b. If stiction is present the isocline equation becomes

$$\dot{E} = -\frac{KE}{MJ+f} - \frac{S\ \mathrm{Sgn}(\dot{E})}{MJ+f}$$

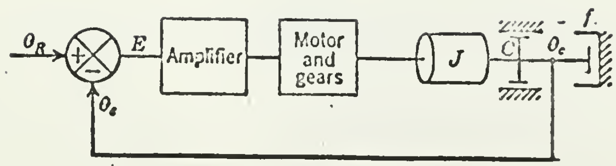
This moves the focal point to $-\frac{S}{K}$. (See Fig. 3-7(b)). A small initial disturbance (such as might be represented at point A) would not produce sufficient torque to pull the system free from the stiction. For a large disturbance (such as A') the system does pull free and begins to acquire a velocity.

c. When a ramp input is applied and coulomb friction is present but stiction may be neglected, the isocline equation becomes

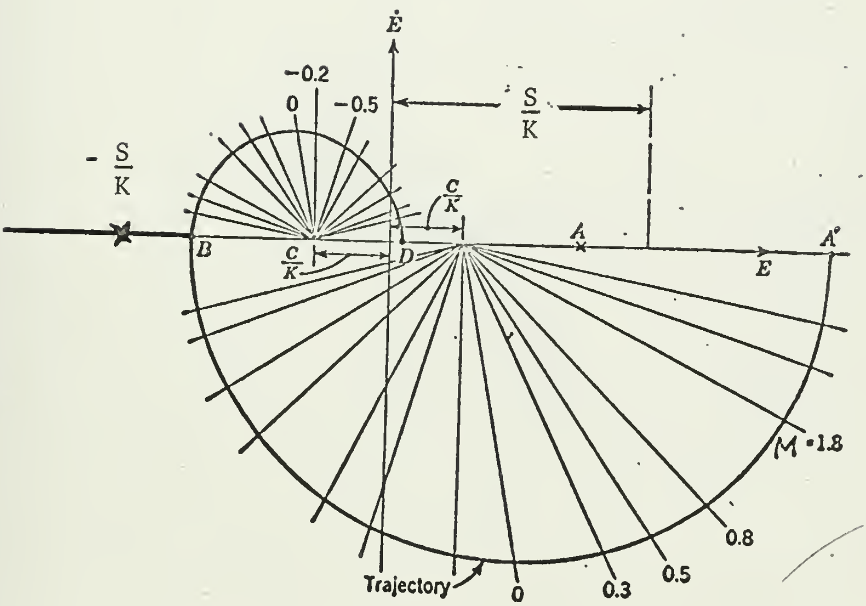
$$\dot{E} = -\frac{KE}{MJ+f} + \frac{f\omega_i + C\ \mathrm{Sgn}(\dot{\theta}_R - \dot{E})}{MJ+f}$$

where $\omega_i = \theta_R$ is the input ramp velocity and the coulomb friction is designated by $C\ \mathrm{Sgn}(\dot{\theta}_R - \dot{E}) = C\ \mathrm{Sgn}(\dot{\theta}_C)$ because the coulomb friction effect is controlled by the relative velocity between the friction surfaces at the output.

The focus is plit by the $C\ \mathrm{Sgn}(\dot{\theta}_R - \dot{E})$ term, and is translated by both C and $f\omega_i$. The foci thus appear at $E = (f\omega_i \mp C)/K$; they are displaced $\pm\, C/K$ from the point $\frac{f\omega_i}{K}$.



(a)



(b)

Fig. 3-7. Phase trajectory for an instrument servo with Coulomb friction and stiction: Step response.

The dividing line is a horizontal straight line at

$$\dot{E} = \omega_i$$

Note:

$$\dot{\theta}_R - \dot{E} = \dot{\theta}_C$$

at

$$\dot{\theta}_C = 0\ , \quad \dot{E} = \dot{\theta}_R = \omega_i$$

(See Fig. 3-8(a)). For a ramp input suddenly applied at $t = 0$, the initial conditions are $E(0)=0, \dot{E}(0)= \omega_i$, which correspond to point P on the figure.

d. When stiction is present but discontinuous, the isocline system of Fig. 3-8(a) is not altered. But the phase trajectory cannot break free from the dividing line at $E = \frac{C}{K}$. It must remain on the dividing line until $E = \frac{S}{K}$. (See Fig. 3-8(b)).

C. APPLICATIONS OF THE ISOCLINE METHOD

Since there are many methods of solution of differential equations which are comparatively simple on the phase-plane but difficult or impossible otherwise, it is rather obvious that the phase-plane provides a very convenient means of analysis of extremely nonlinear first and second-order differential equations. It seems that the isocline method is a general and useful method with which to approach problems by using the phase-plane method. Application of the isocline method to specific problems illustrates the usefulness of this method.

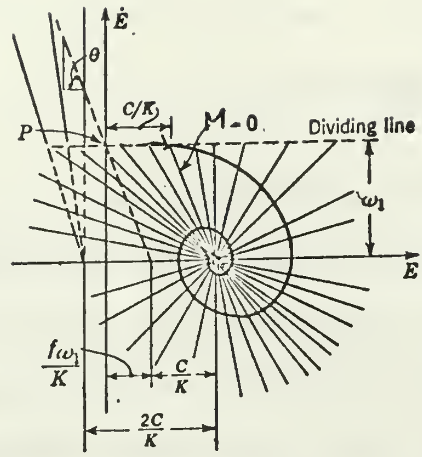


Fig. 3-8a. Phase portrait of an instrument servo with Coulomb friction in response to a ramp input.

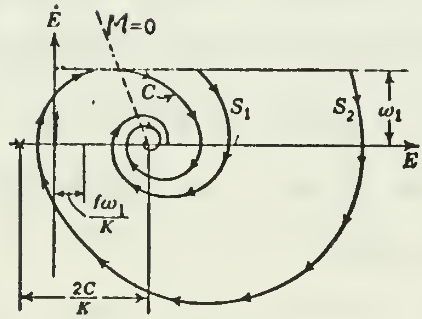


Fig. 3-8b. Effect of stiction on the ramp response of a servo with Coulomb friction.

1. First-Order System

The first-order equation has the form

$$\frac{dx}{dt} = f(x,t) \tag{3-1}$$

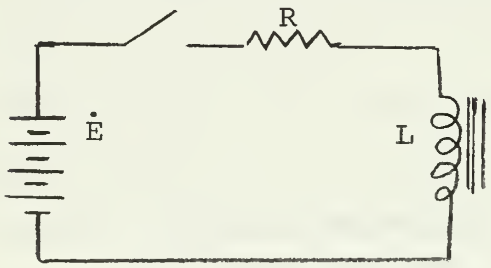
Assume that the function f(x,t) is continuous and single-valued with the possible exception of certain singular points. The isocline method requires a graphical construction performed on axes of x and t and yields a solution as a curve. For any given point on the x-t plane, the numerical value of f(x,t), and thus of $\frac{dx}{dt}$, can be calculated.

Example 1: Find a solution for the equation

$$\frac{dx}{dt} = -x + t$$

Consider the initial conditions $x_o = 0$ at $t_o = 0$, and also $x_o = 0.2$ at $t_o = 0$. For this linear equation, the isoclines are straight lines given by the relation m = x + t. A family of isoclines, each carrying its line segments of proper slope is shown in Fig. 3-9. Solution curves are sketched in the figure, starting at each of the specified initial conditions.

Example 2: RL circuit with nonlinear L



$$L\frac{di}{dt} + iR = Eu(t)$$

Since the inductor is nonlinear, and the coefficient L therefore variable, it is more convenient to express the relationship as

$$M \frac{d\phi}{dt} + iR = Eu(t)$$

from which

$$\frac{d\phi}{dt} = \frac{Eu(t) - iR}{M}$$

The equation of the isoclines is obtained by letting $\frac{d\phi}{dt} = M$, some constant, and noting that $Eu(t) = E$ for $t > 0$, then

$$i = \frac{E - MM_1}{R}$$

Knowing the magnetization curve of the inductor, one can obtain the transient current variation of the nonlinear RL circuit, i vs t. (See Fig. 3-10)

2. Second-order System

(Example 1) The equation for a damped pendulum is:

$$\frac{d^2x}{dt} + K \frac{dx}{dt} + \sin x = 0$$

Let

$$v = \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

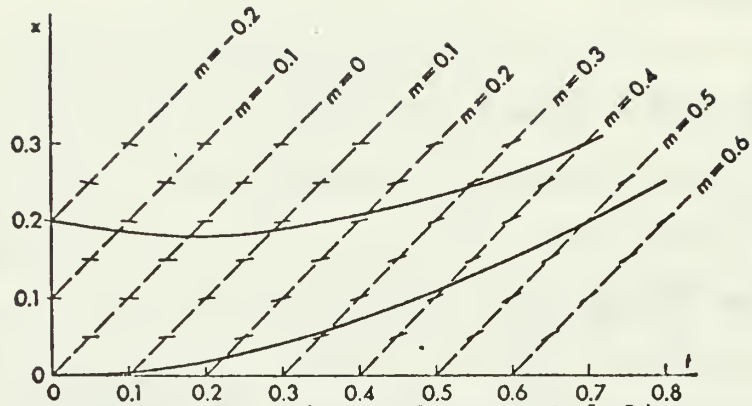


Fig. 3-9. (Example III-C-1-A).

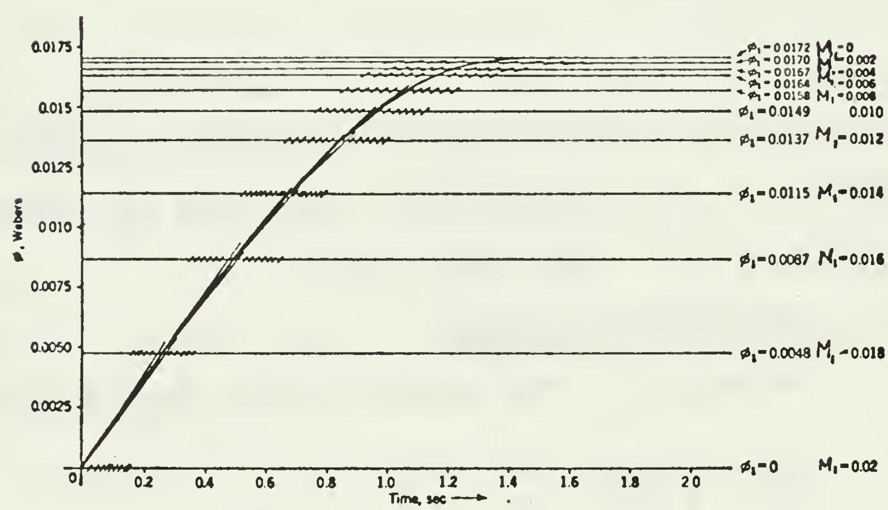


Fig. 3-10. Example (III-C-1-B).

Thus

$$\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \sin x$$

$$= \frac{dv}{dx} + Kv + \sin x = 0$$

or

$$\frac{dv}{dx} = -\frac{(Kv + \sin x)}{v}$$

Let

$$K = 1$$

$$\frac{dv}{dx} = -\frac{(v + \sin x)}{v}$$

$\frac{dv}{dx}$ is the slope at any point x, v of a possible phase-plane trajectory.

Now assume a number of values for $\frac{dv}{dx}$

$\frac{dv}{dx}$	Equation of v vs x
0	$v = -\sin x$
1	$v = -\frac{1}{2}\sin x$
-1	$0 = -\sin x$
2	$v = -\frac{1}{3}\sin x$
-2	$v = \sin x$
3	$v = -\frac{1}{4}\sin x$
-3	$v = \frac{1}{2}\sin x$

(See Fig. 3-11).

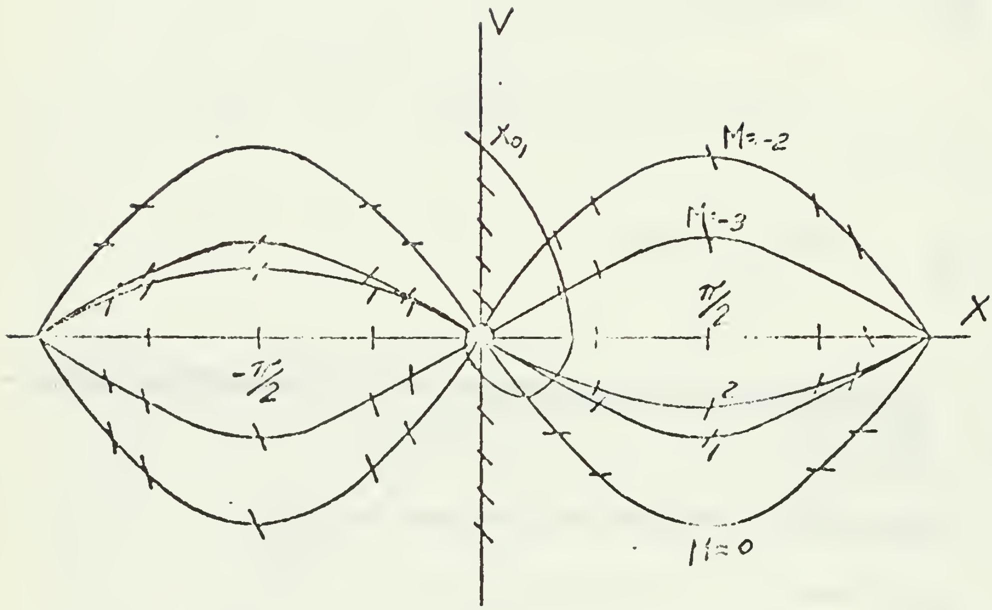


Fig. 3-11. Isocline solution.

Example 2: Nonlinear system $\ddot{x} + x\dot{x} + x = 0$

Let $\ddot{x} = M\dot{x}$;

then

$$M\dot{x} + x\dot{x} + x = 0$$

$$\dot{x} = \frac{-x}{M + x}$$

Draw a number of straight lines $\ddot{x} = M\dot{x}$ through the origin (corresponding to different values of M) on the $\ddot{x}$-$\dot{x}$ plane (as in Fig. 3-12(a)).

Plot $\ddot{x}$ vs $\dot{x}$ from equation $\ddot{x} + x\dot{x} + x = 0$ for constant (different) values of x (see Fig. 3-12(a)).

With any particular straight line in view, say $M = M_1$, determine all possible points of intersections of above plots. The procedure is repeated for $M = M_2$, M_3, etc., until a fair number of isocline curves are obtained on the phase plane. The respective slope lines are marked on these isoclines. Joining these slope lines gives the different trajectories or the entire phase portrait (as in Fig. 3-12(b)).

Example 3: Consider the case of a saturating electronic amplifier driving a motor in an instrument servo. (See Fig. 3-13(a))

The equation in the linear region is

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} + \frac{K_s K K_m}{N\tau_m} E = 0 \qquad -E_1 < E < +E_1$$

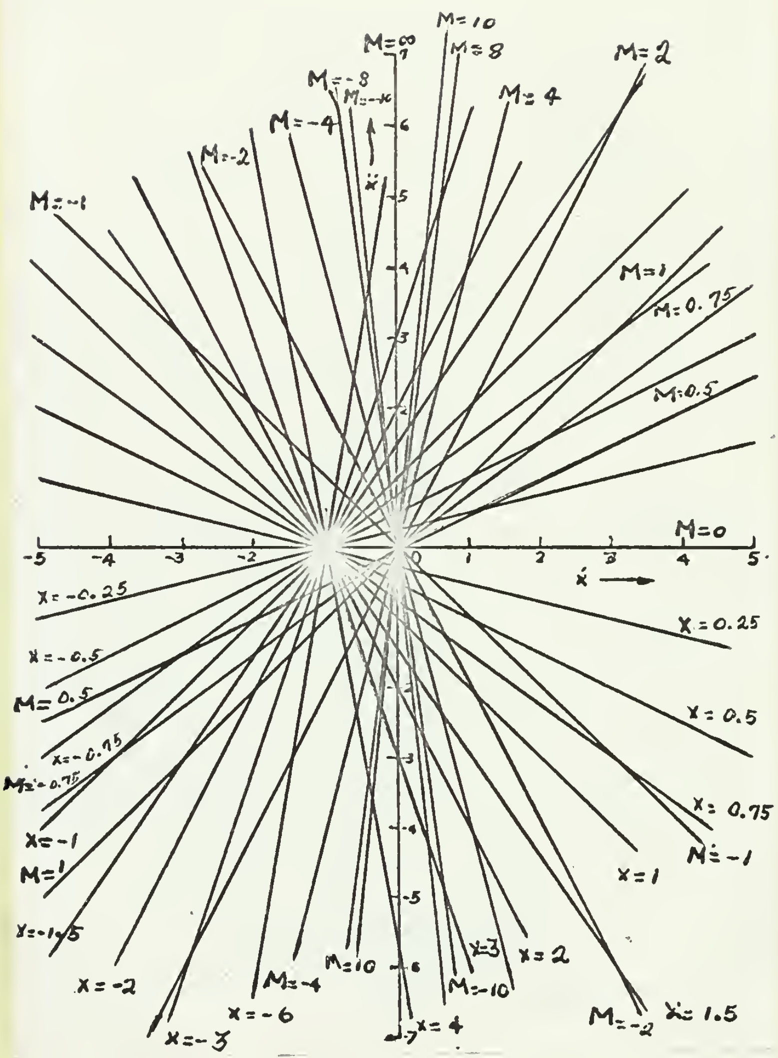


Fig. 3-12(a) Example $\ddot{x} + x\dot{x} + x = 0$

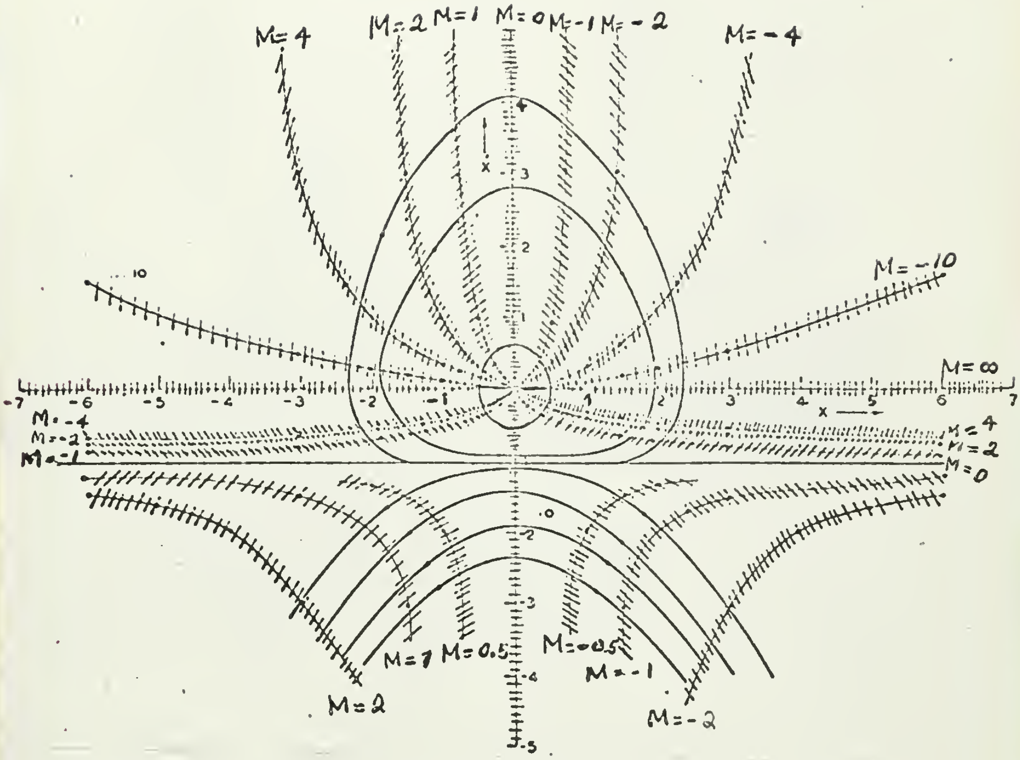


Fig. 3-12(b). Example $\ddot{x} + x\dot{x} + x = 0$

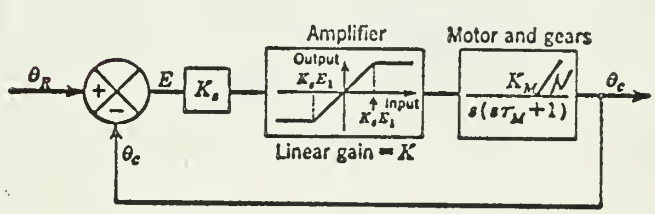


Fig. 3-13(a)

where E_1 is the value of $|E|$ which causes saturation and N is the gear ratio.

For saturation operation, the equations are;

$$\ddot{E} + \frac{1}{\tau_m}\dot{E} = +C \qquad E \le -E_1$$

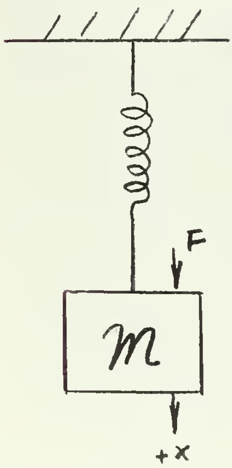
$$\ddot{E} + \frac{1}{\tau_m}\dot{E} = -C \qquad E \ge +E_1$$

where

$$C = \frac{K_s K K_m}{N\tau_m} E_1$$

On the E vs $\dot{E}$ phase plane the isoclines are all straight lines; in the linear region the isoclines are radial straight lines emanating from a focus at the origin. While in the saturation region the isoclines are horizontal, the phase trajectory is constructed in the usual manner (as in Fig. 3-13(b)).

Example 4: Oscillator with relay control.



$$m\ddot{x} + Kx = F$$

$$\ddot{x} + \frac{K}{m}x = \frac{F}{m}$$

Let

$$\frac{K}{m} = 1$$

$$\frac{F}{m} \triangleq u$$

then $\ddot{x} + x = u$

Let

$$\dot{x} = y$$

then

$$\ddot{x} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = M\dot{x}$$

where

$$M = \frac{dy}{dx}$$

thus

$$My + x = u$$

$$M = \frac{u - x}{y}$$

a. If $u = K$ the phase trajectory is as in Fig. 3-14(a).

b. If $u = -K\ \mathrm{Sgn}(x)$

$$M = -\frac{x + K\ \mathrm{Sgn}(x)}{y}$$

$x = 0$ or y-axis is a dividing line. (See Fig. 3-14(b))

If x is positive $M = -\dfrac{x + K}{y}$

if x is negative $M = -\dfrac{x - K}{y}$

c. If $u = -K\ \mathrm{Sgn}(y)$

$$M = -\frac{x + K\ \mathrm{Sgn}(y)}{y}$$

$y = 0$ or x-axis is a dividing line. (See Fig. 3-14(c))

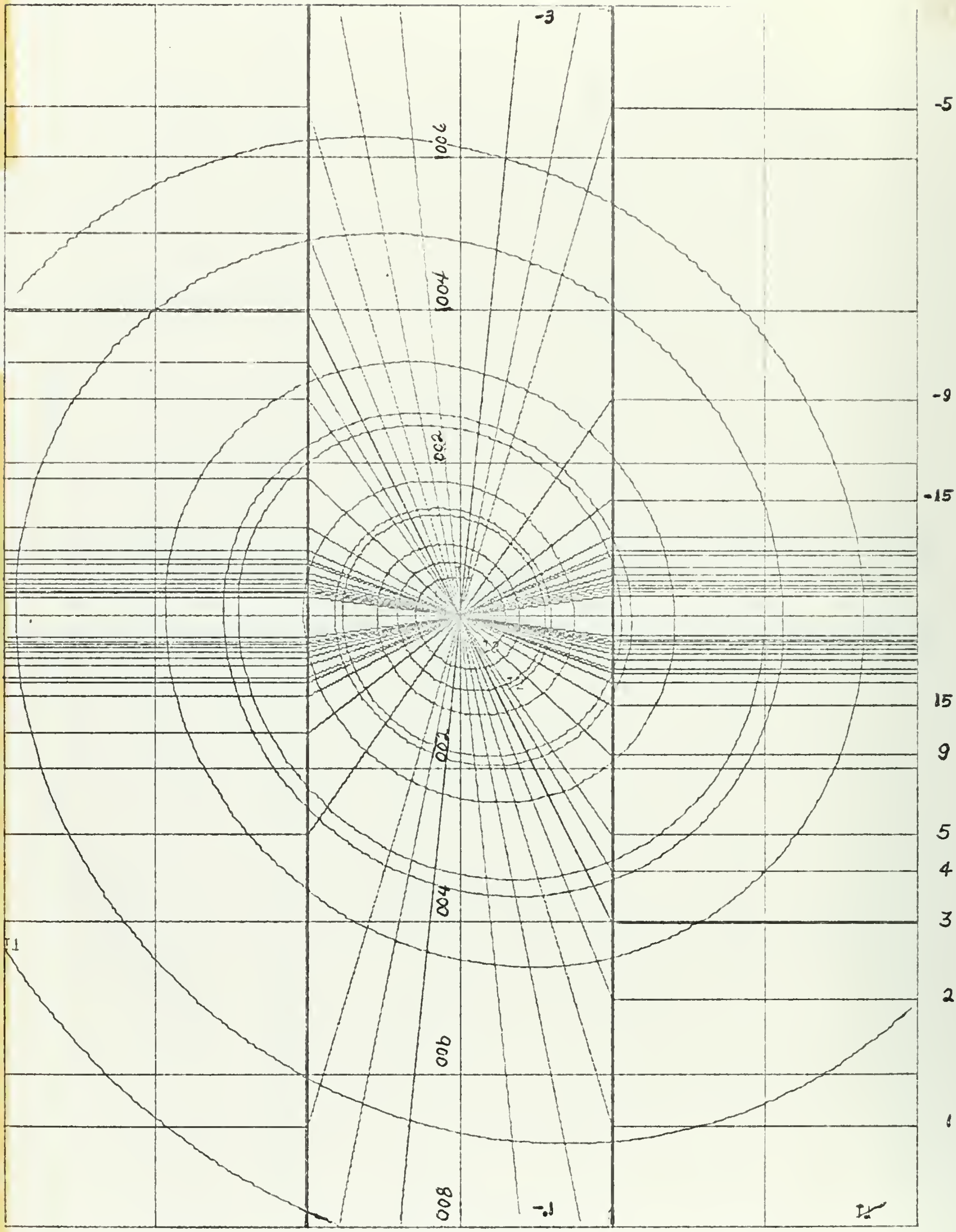


Fig. 3-13(b)

X-Scale = 2.00E-01 Units Inch.

Y-Scale = 2.00E-01 Units Inch.

If y is positive $\quad M = -\dfrac{x + K}{y}$

If y is negative $\quad M = -\dfrac{x - K}{y}$

c. If $u = -K\,\mathrm{Sgn}(x+y)$

$$M = \frac{x - K\,\mathrm{Sgn}(x+y)}{y}$$

$x + y = 0$ is a dividing line.

(See Fig. 3-14(d))

e. If $u = -K\,\mathrm{Sgn}(x-y)$

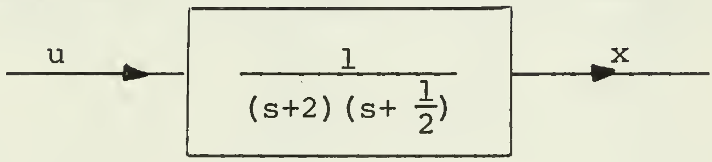
$$M = \frac{x - K\,\mathrm{Sgn}(x-y)}{y}$$

$x - y = 0$ is a dividing line.

(See Fig. 3-14(e))

Due to the changing dividing line the system became unstable.

Example 5:



Given:

$$u = \begin{cases} 0 \quad , & |x| < 0.5 \\ -\mathrm{Sgn}(x), & |x| \geq 0.5 \end{cases}$$

Since

$$\frac{x(s)}{u(s)} = \frac{1}{(s+2)(s+\frac{1}{2})} = \frac{1}{s^2 + \frac{5}{2}s + 1}$$

$$\ddot{x} + \frac{5}{2}\dot{x} + x = u$$

Let

$$\dot{x} = y$$

then

$$\ddot{x} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = M\dot{x}$$

thus

$$M = \frac{-\frac{5}{2}y - x + u}{y}$$

At $u = 0$, the equation of isoclines is:

$$My = -\frac{5}{2}y - x$$

At $u = -\text{Sgn}(x)$

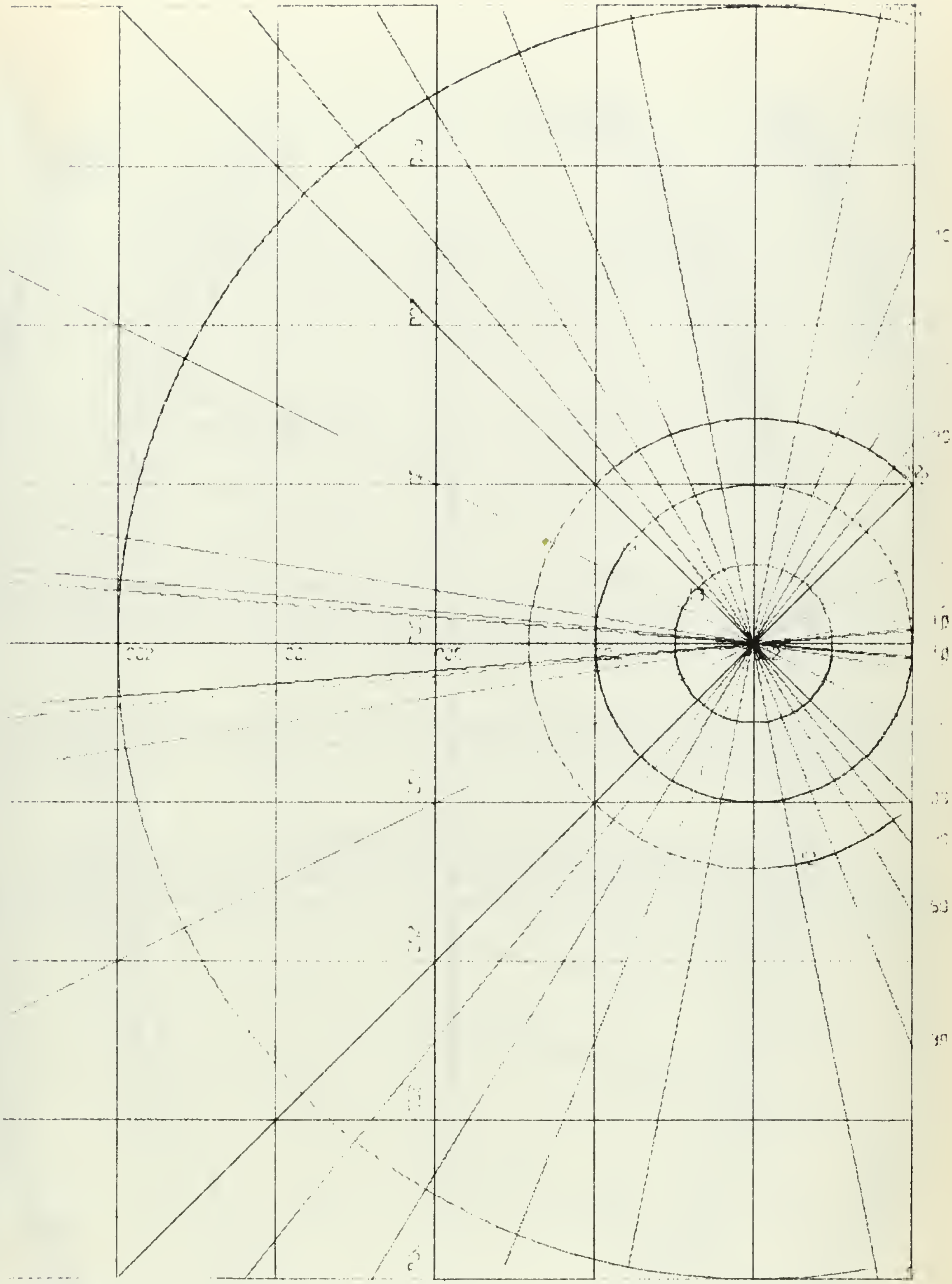
If x is positive, the equation of isoclines is

$$My = -\frac{5}{2}y - x - 1$$

If x is negative, the equation of isoclines is

$$My = -\frac{5}{2}y - x + 1$$

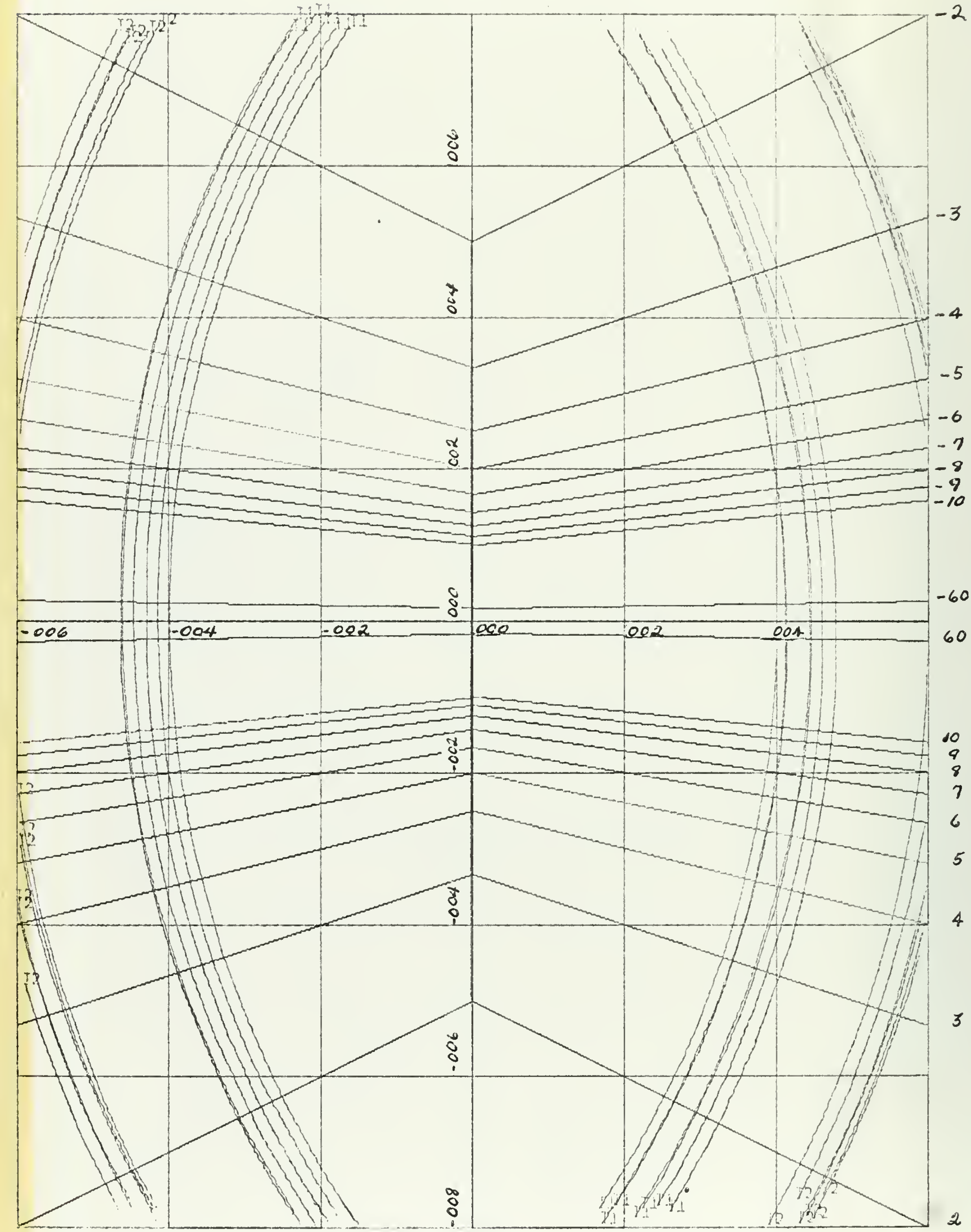
(See Fig. 3-15).



X-Scale = 1.00E+00 Units Inch.

Y-Scale = 1.00E+00 Units Inch.

Fig. 3-14(a)



X-Scale = 2.00E-01 Units Inch.

Y-Scale = 2.00E-01 Units Inch.

Fig. 3-14(b)

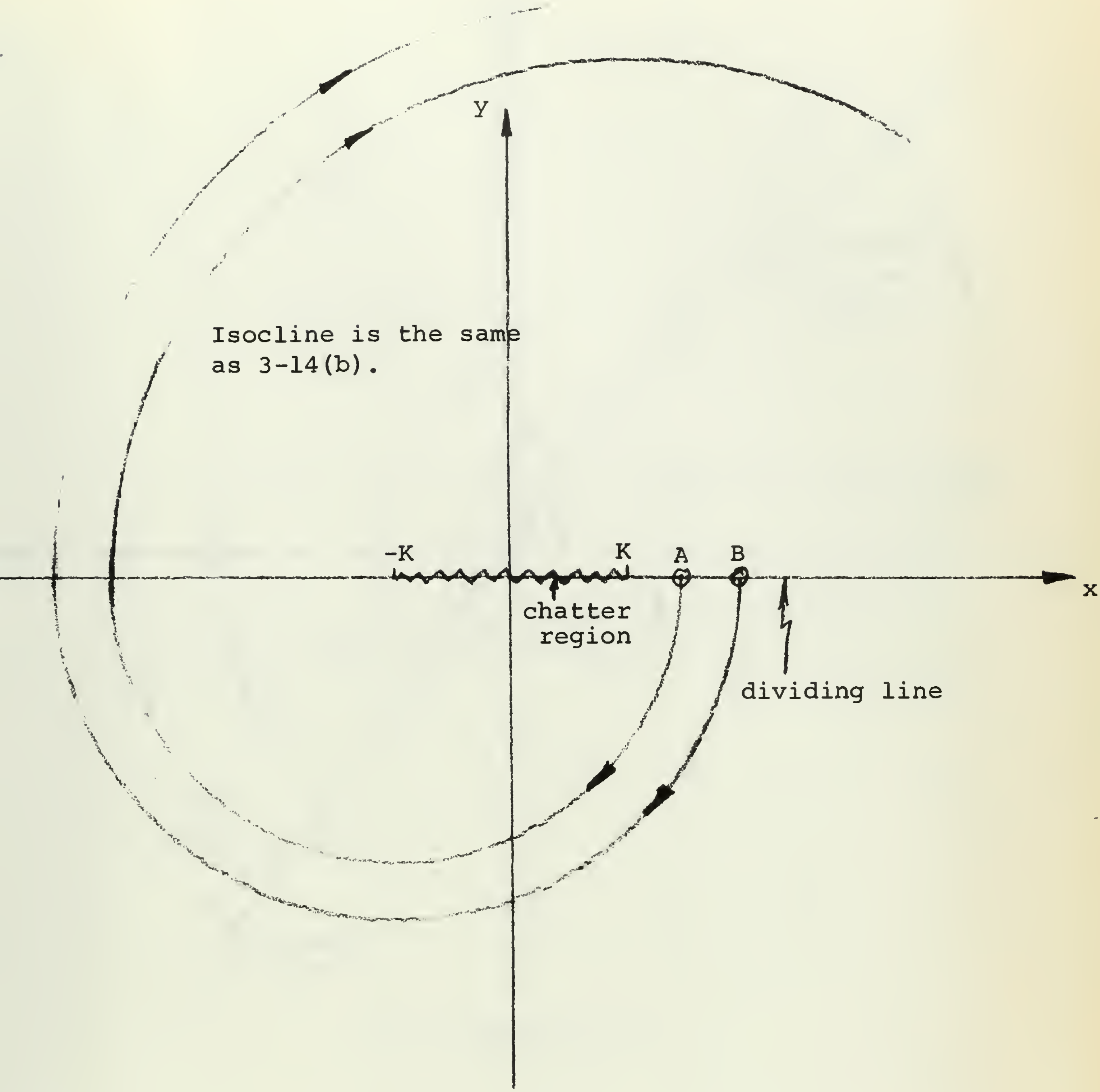

y
Isocline is the same
as 3-14(b).
-K
K
A
B
x
chatter
region
dividing line

Fig. 3-14(c).

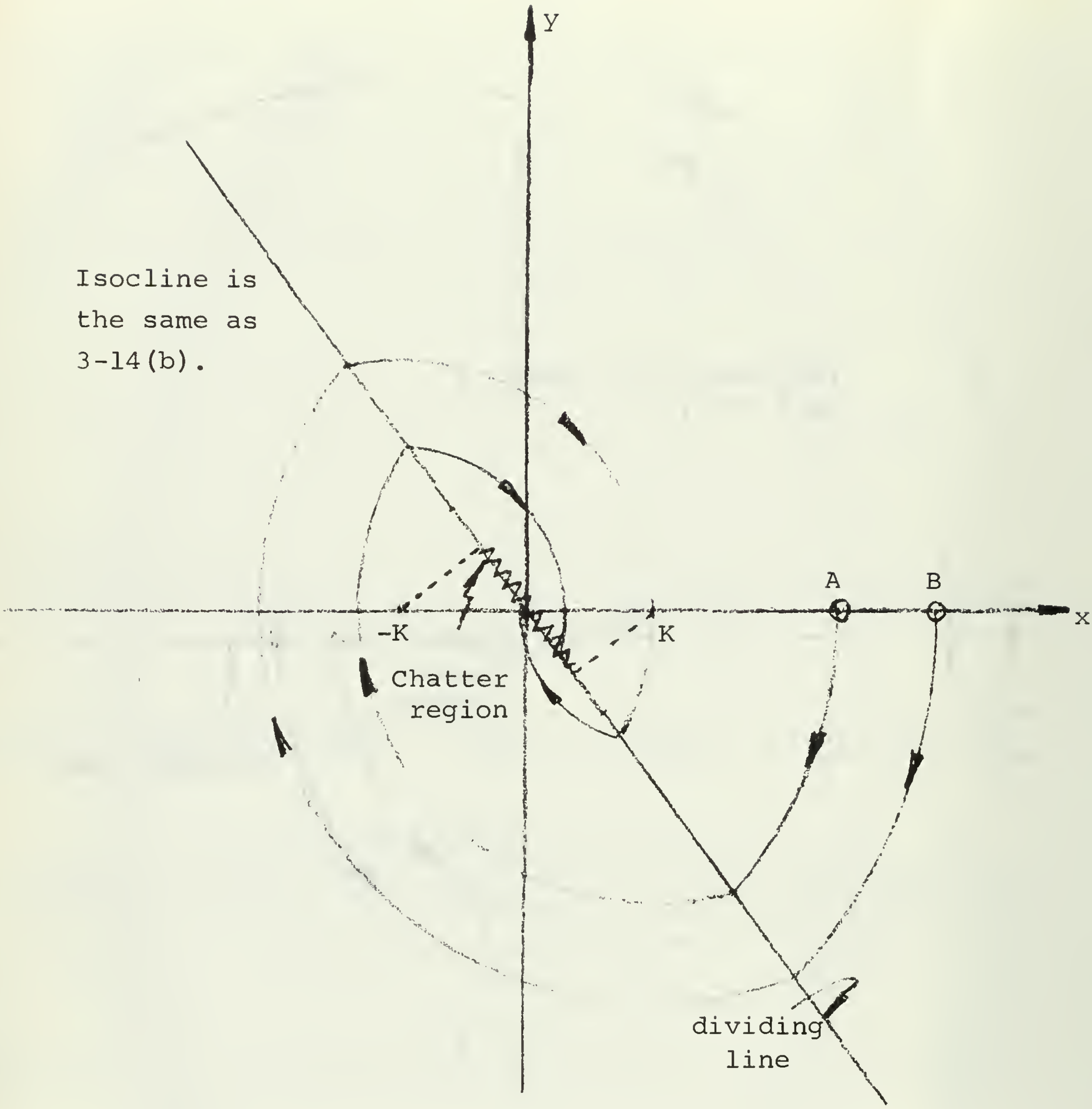

y
Isocline is
the same as
3-14(b).
A
B
x
-K
K
Chatter
region
dividing
line

Fig. 3-14(d)

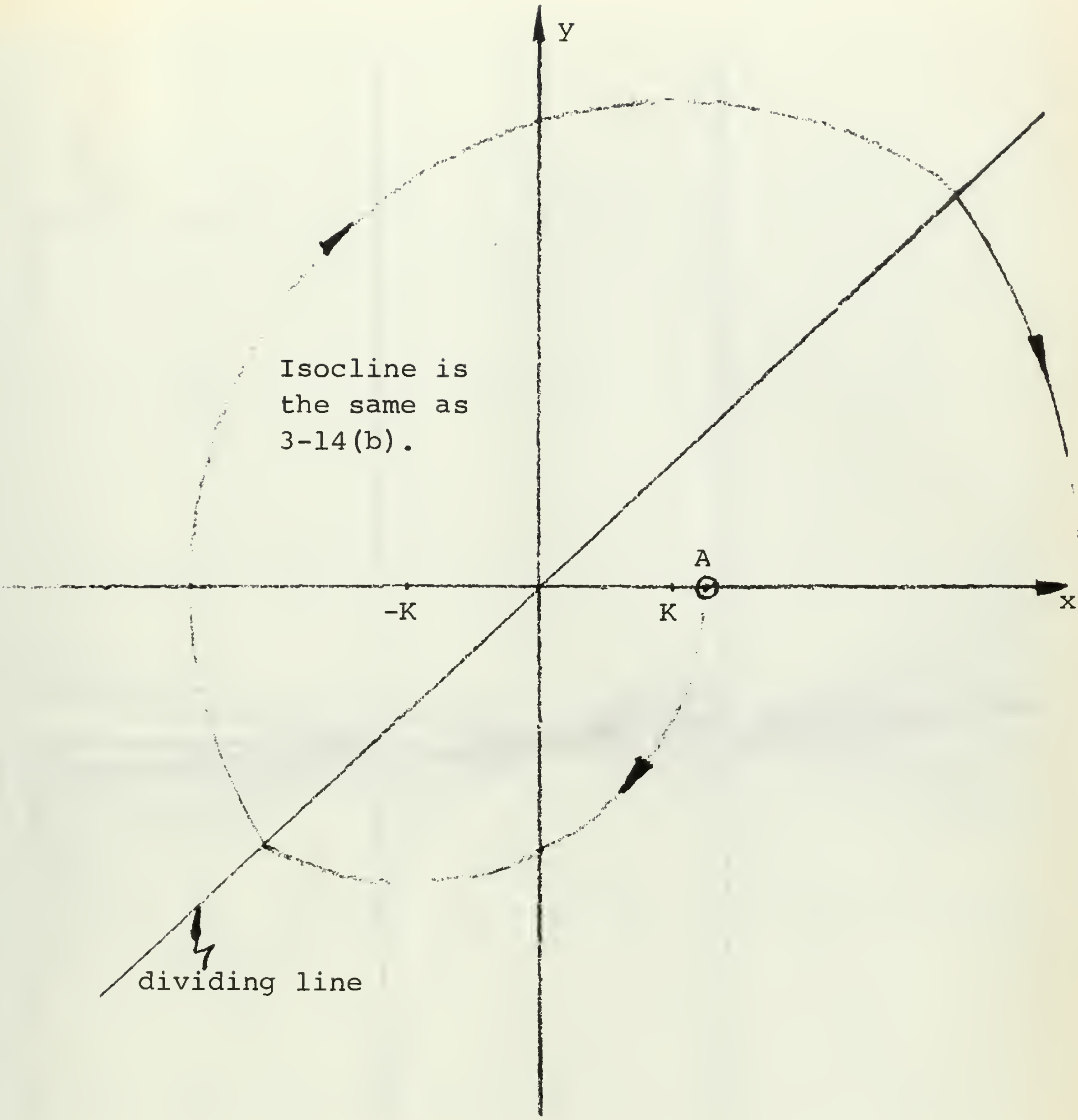


Fig. 3-14(e)

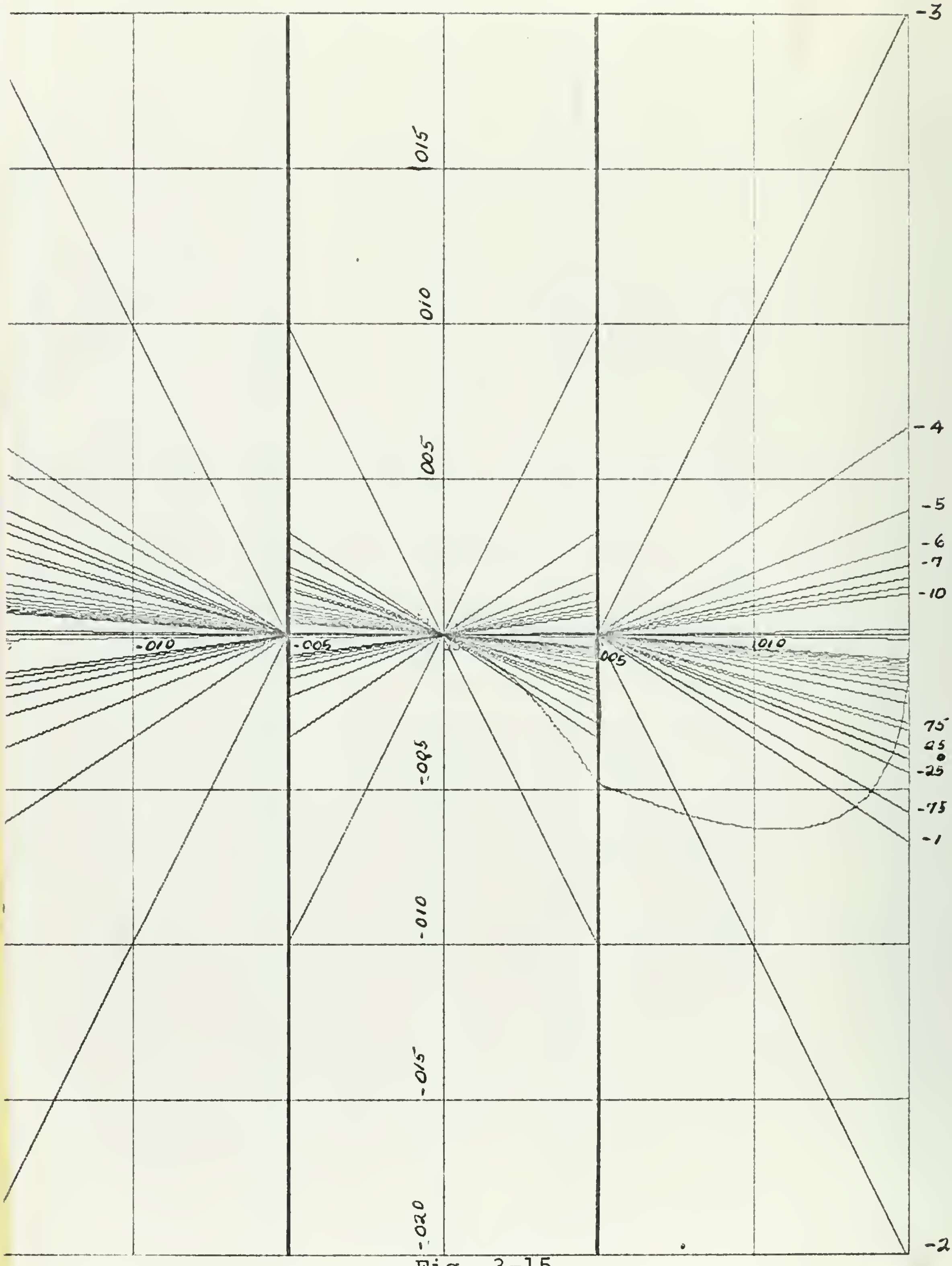


Fig. 3-15.

X-Scale = 5.00E-01 Units Inch.

Y-Scale = 5.00E-01 Units Inch.

3. High-Order Systems

The phase-plane, as normally used, is restricted to problems which can be described by first and second-order differential equations. This is a fairly serious restriction, since many practical systems require higher-order differential equations for an adequate description. In the discussion of numerical methods for solving differential equations, first-order equations are studied in the beginning. Then equations of higher order are solved by replacing each equation of nth order by n equations of first-order and solving these simultaneously. In like manner, the discussion of graphical methods has begun by considering the isocline method of solving a first-order equation first, then second-order equations. But further consideration may not be possible since there is not enough information available initially to construct isocline curves for the several first-order equations that would arise from a single high-order equation. So, if we can factor high-order equations into several second-order equations, these can be solved by phase-plane method. The difficulty is in converting a high-order equation into several second-order equations.

No convenient long hand method is available, but use of the digital computer permits study of such phase space problems.

IV. GENERATING THE PROGRAMS TO DRAW ISOCLINES AND TRAJECTORIES OF SECOND-ORDER DIFFERENTIAL EQUATIONS

A. $\ddot{x} + F(x)\ \dot{x} + G(x) = 0$ type equations (See Appendix, Program 1) (Where F(x) f G(x) can be any function of x)

$$\ddot{x} + F(x)\ \dot{x} + G(x) = 0$$

Let

$$y = \dot{x}$$

$$\ddot{x} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Let

$$\frac{dy}{dx} = M$$

then

$$\ddot{x} = My$$

thus

$$\ddot{x} + F(x)\ \dot{x} + G(x)$$

$$= My + F(x)\ y + G(x) = 0$$

$$y = - \frac{G(x)}{M + F(x)} \quad \text{equation of isoclines.}$$

Trajectories:

```
ZDØT(1) =  Z(2)
ZDØT(2) = -F(Z(1))* Z(2) - G(Z(1))
```

(By using subroutine program RKLDEQ.)

Example 1: $\ddot{x} + x\dot{x} + x = 0$

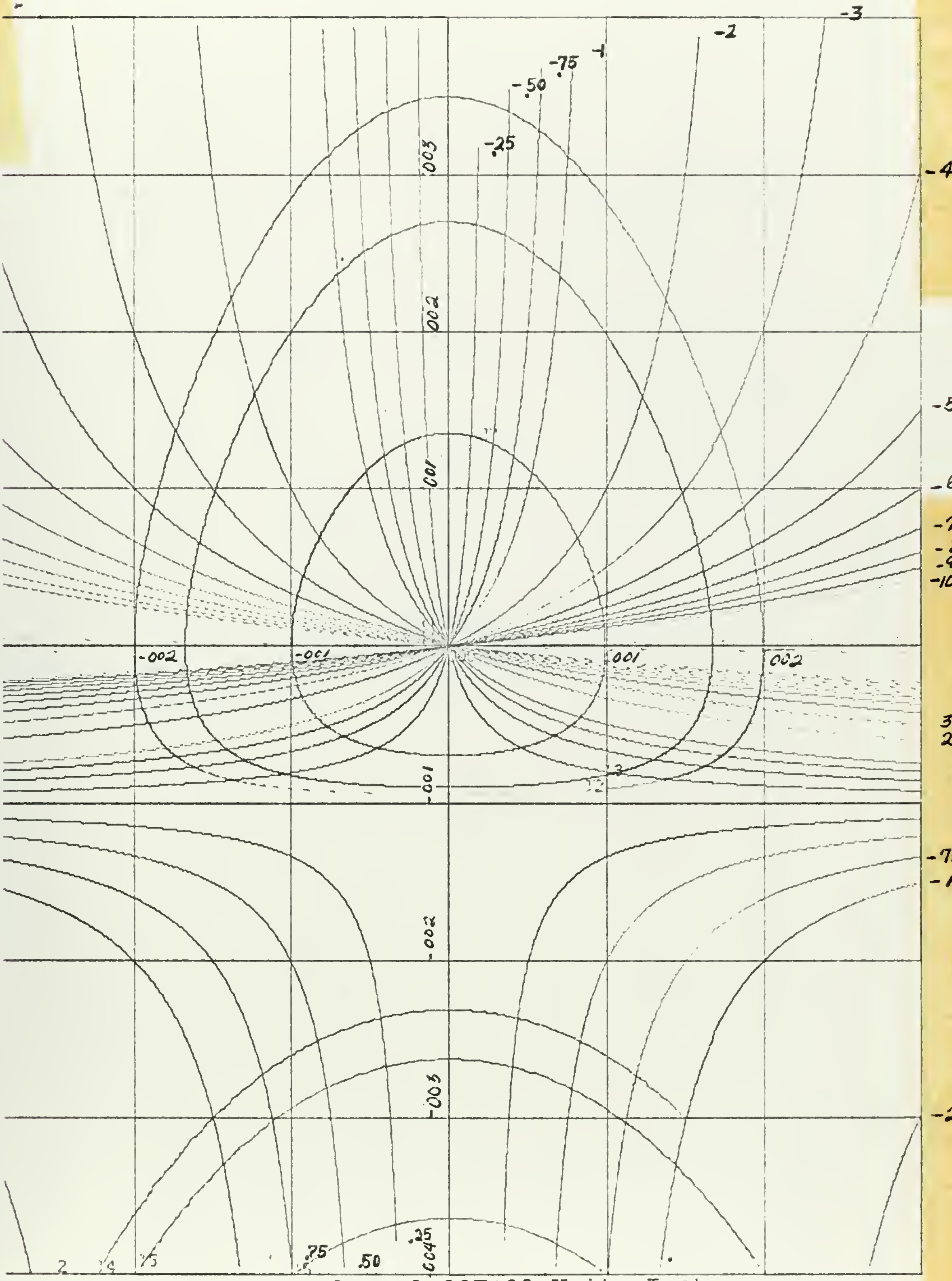


Fig. 4-1. X-Scale = 1.00E+00 Units Inch.

Y-Scale = 1.00E+00 Units Inch.

Example 2: $\ddot{x} - e(1-x^2)\dot{x} + x = 0$

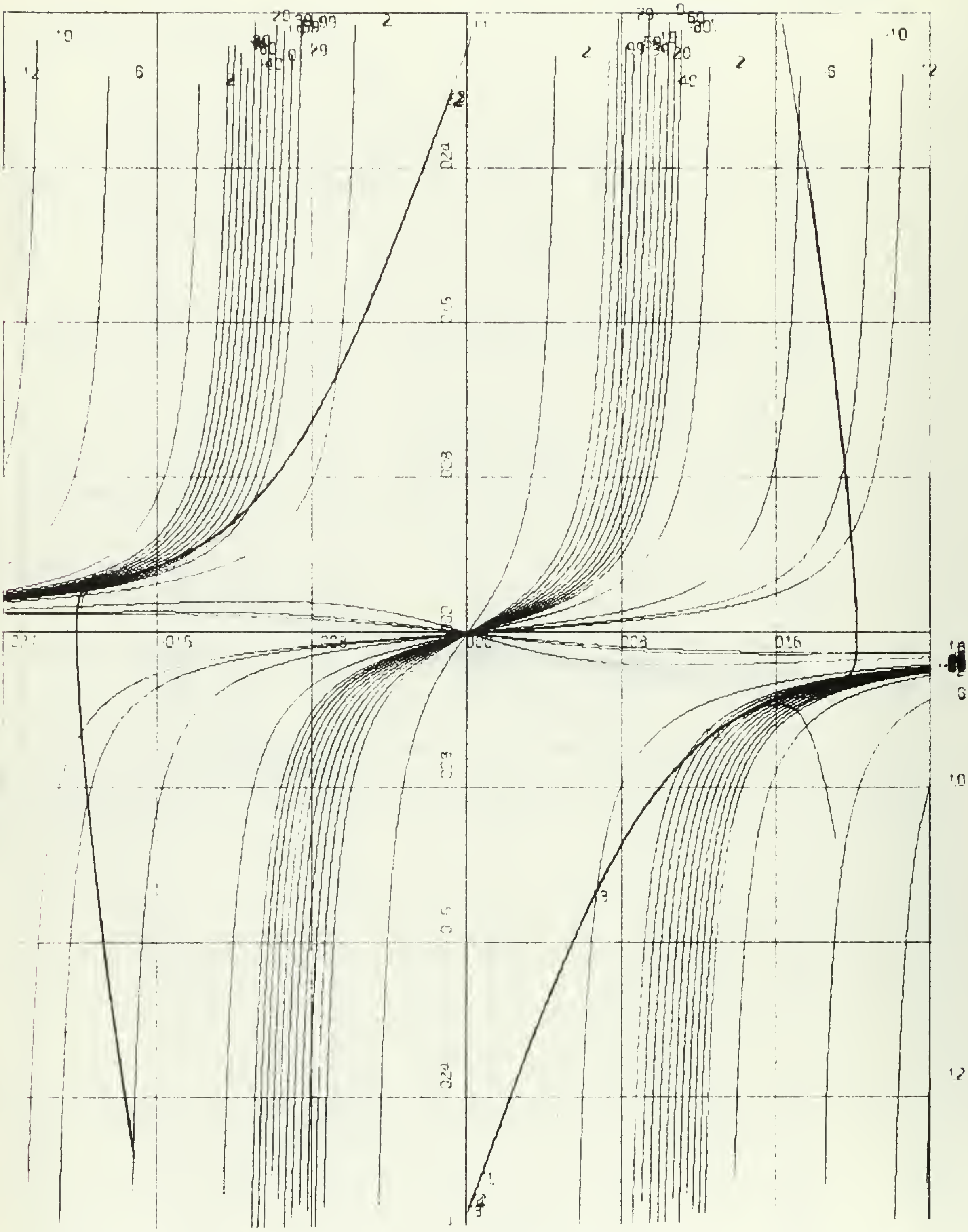


Fig. 4-2. X-Scale = 8.00E-01 Units Inch.
Y-Scale = 8.00E-01 Units Inch.

Example 3: $\ddot{x} + (\cos x)\dot{x} + \tan^2 x = 0$

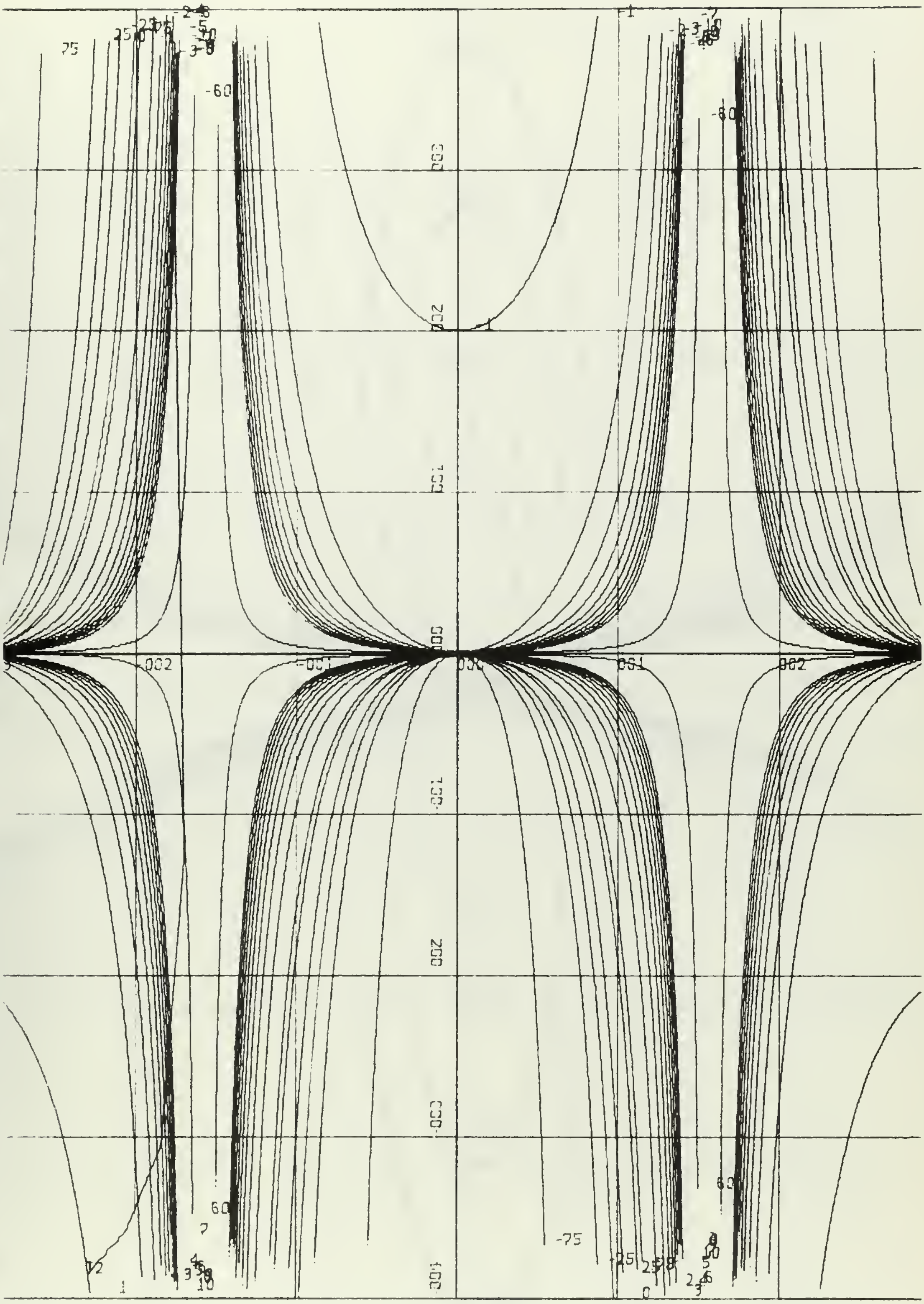


Fig. 4-3. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 4: $\ddot{x} + 25(1 + 0.1x^2) = 0$

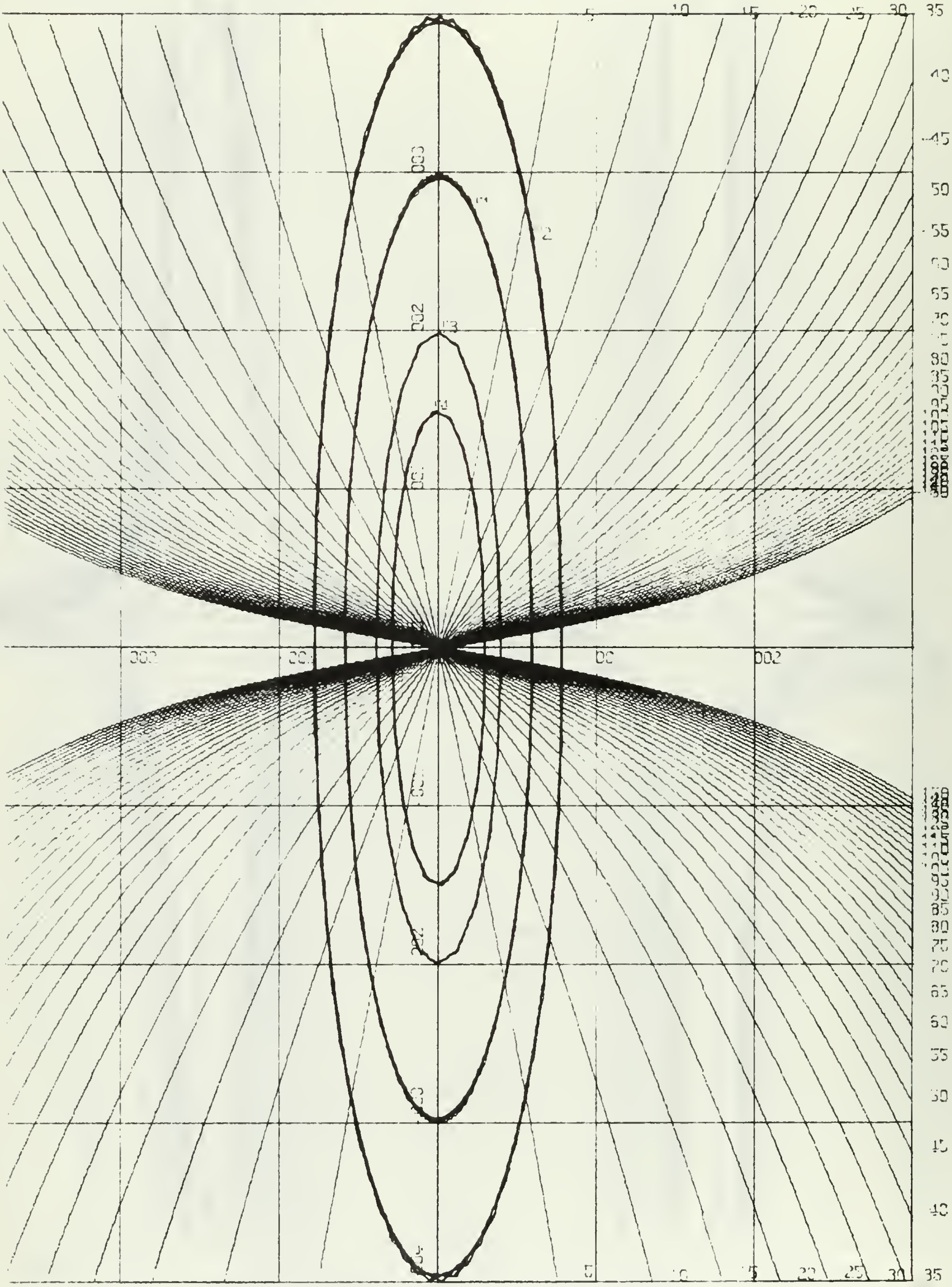


Fig. 4-4. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 5: $\ddot{x} + 0.5\dot{x} + x = 0$

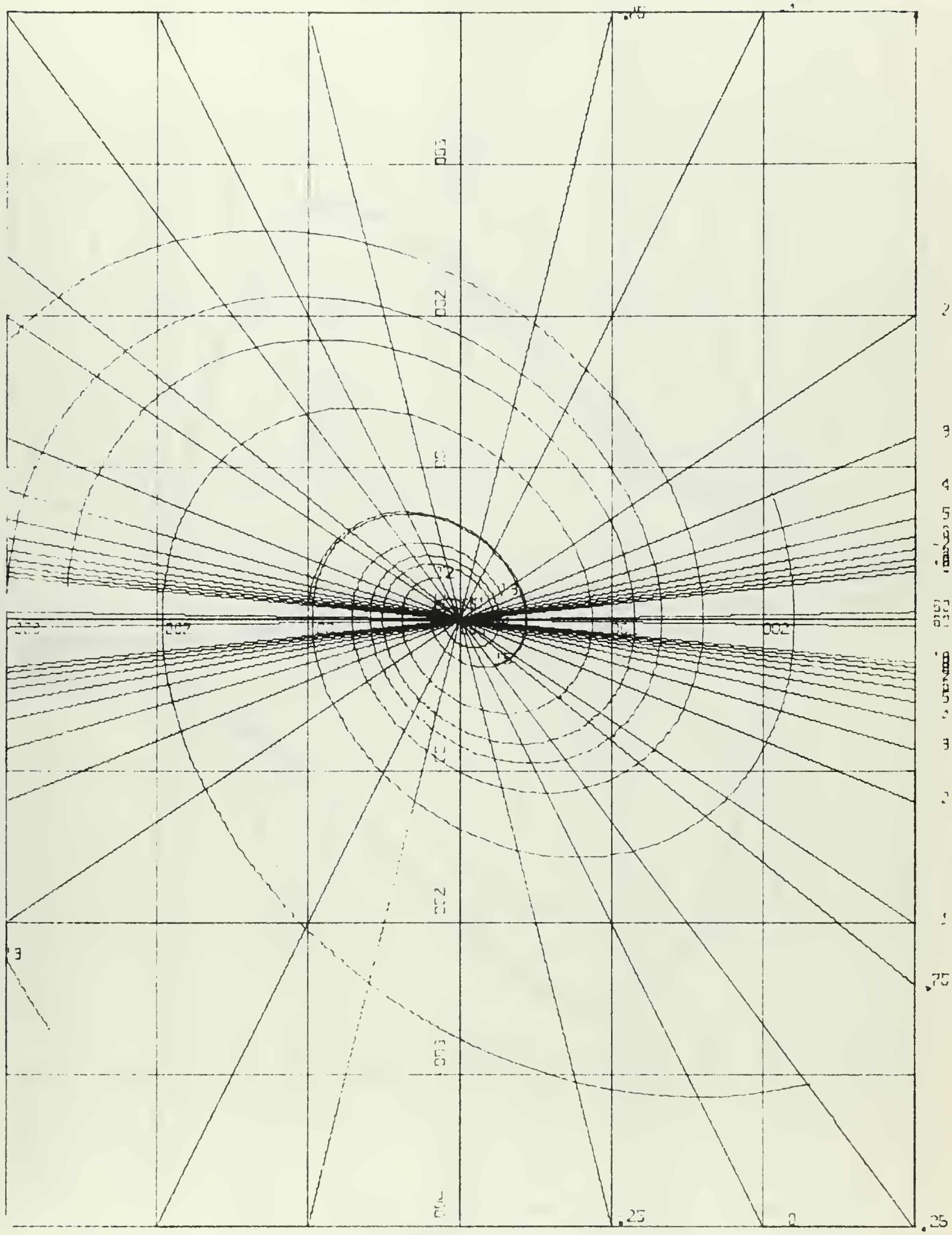


Fig. 4-5. X-Scale = 1.00E+01 Units Inch.
Y-Scale = 1.00E+01 Units Inch.

Example 6: $\ddot{x} - (1-x^2)\dot{x} + x = 0$

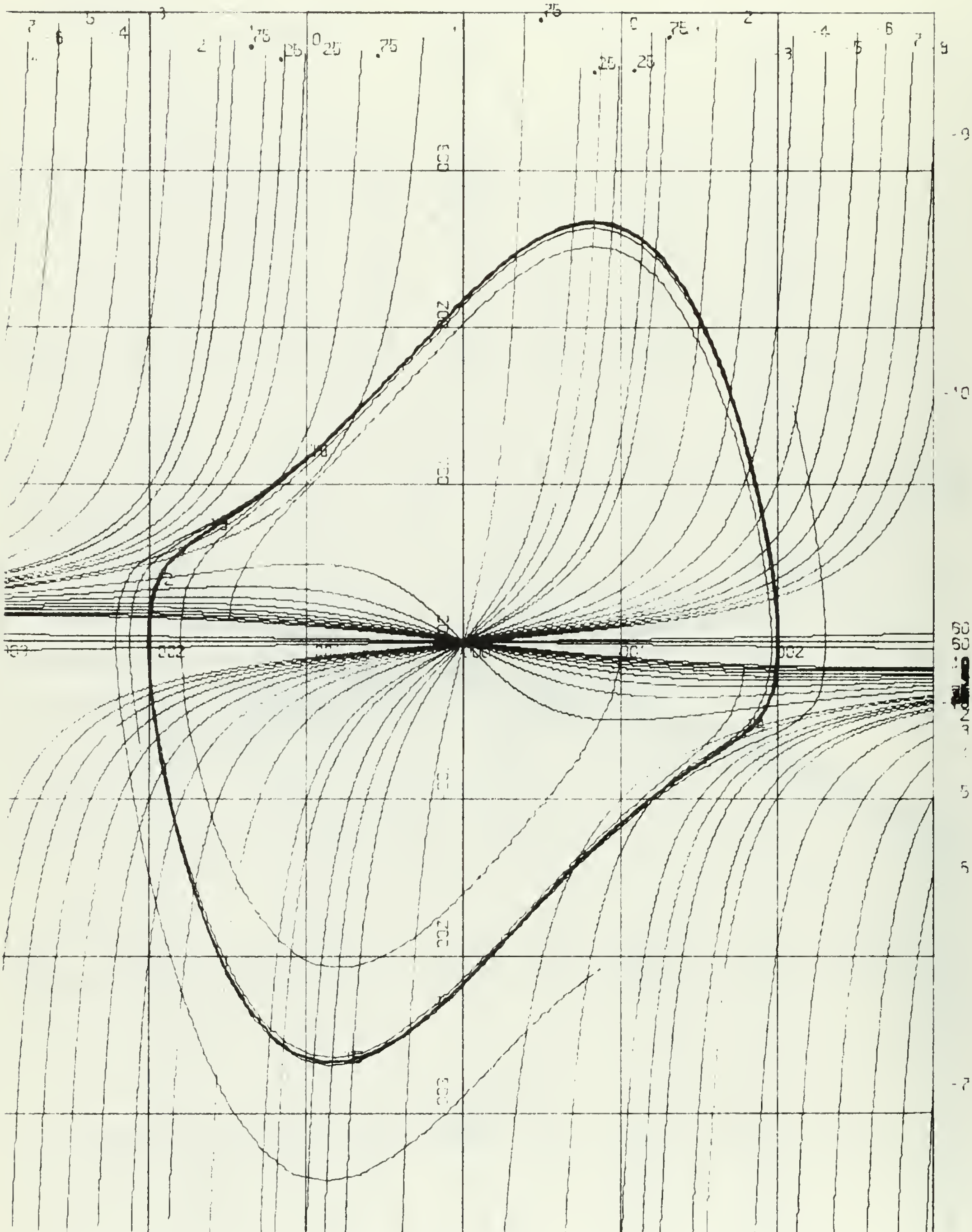


Fig. 4-6. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 7: $\ddot{x} + \dot{x} + x|x| = 0$

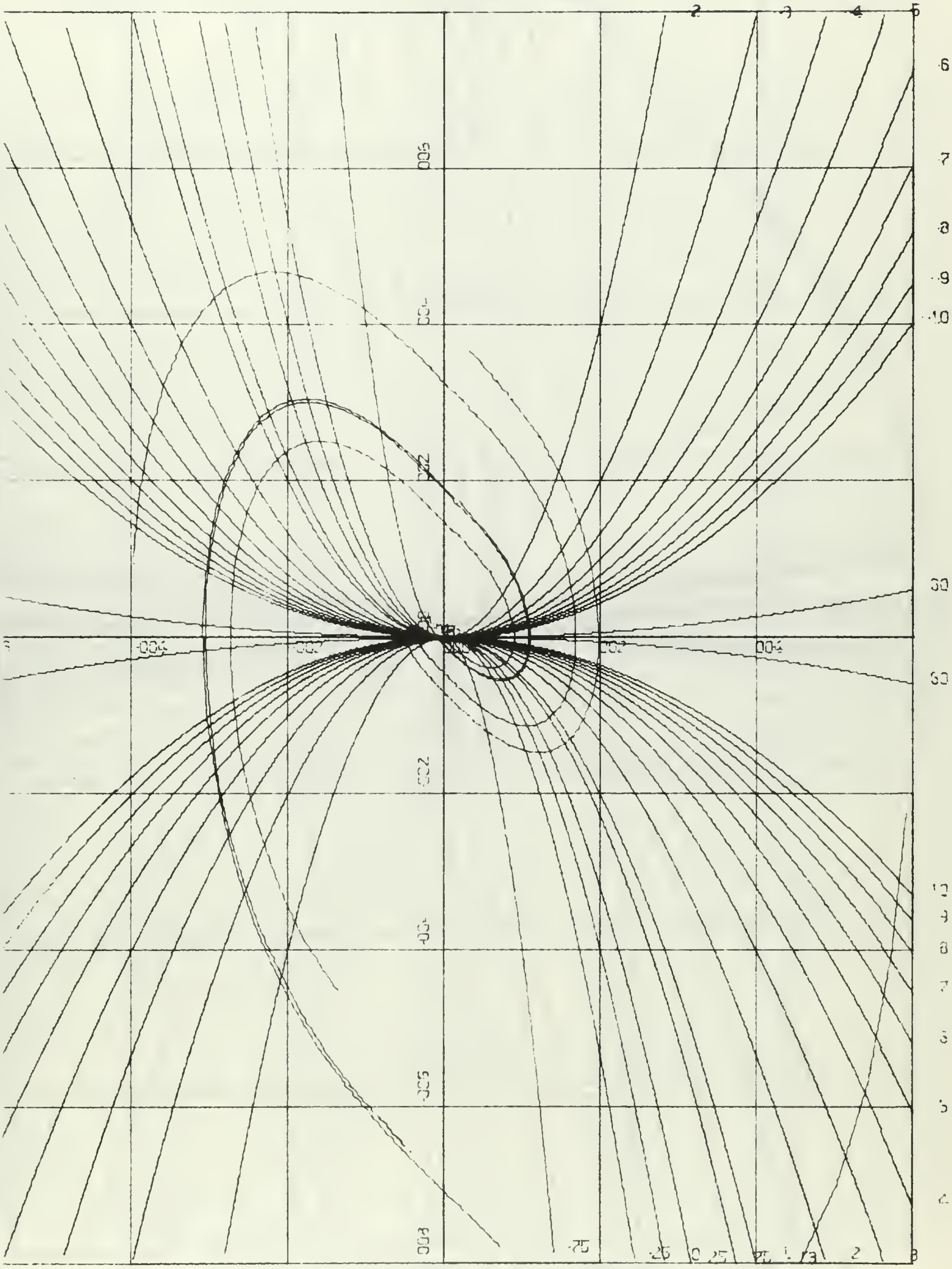


Fig. 4-7. X-Scale = 2.00E+00 Units Inch.
Y-Scale = 2.00E+00 Units Inch.

Example 8: $\ddot{x} + \frac{1}{x}\dot{x} + x = 0$

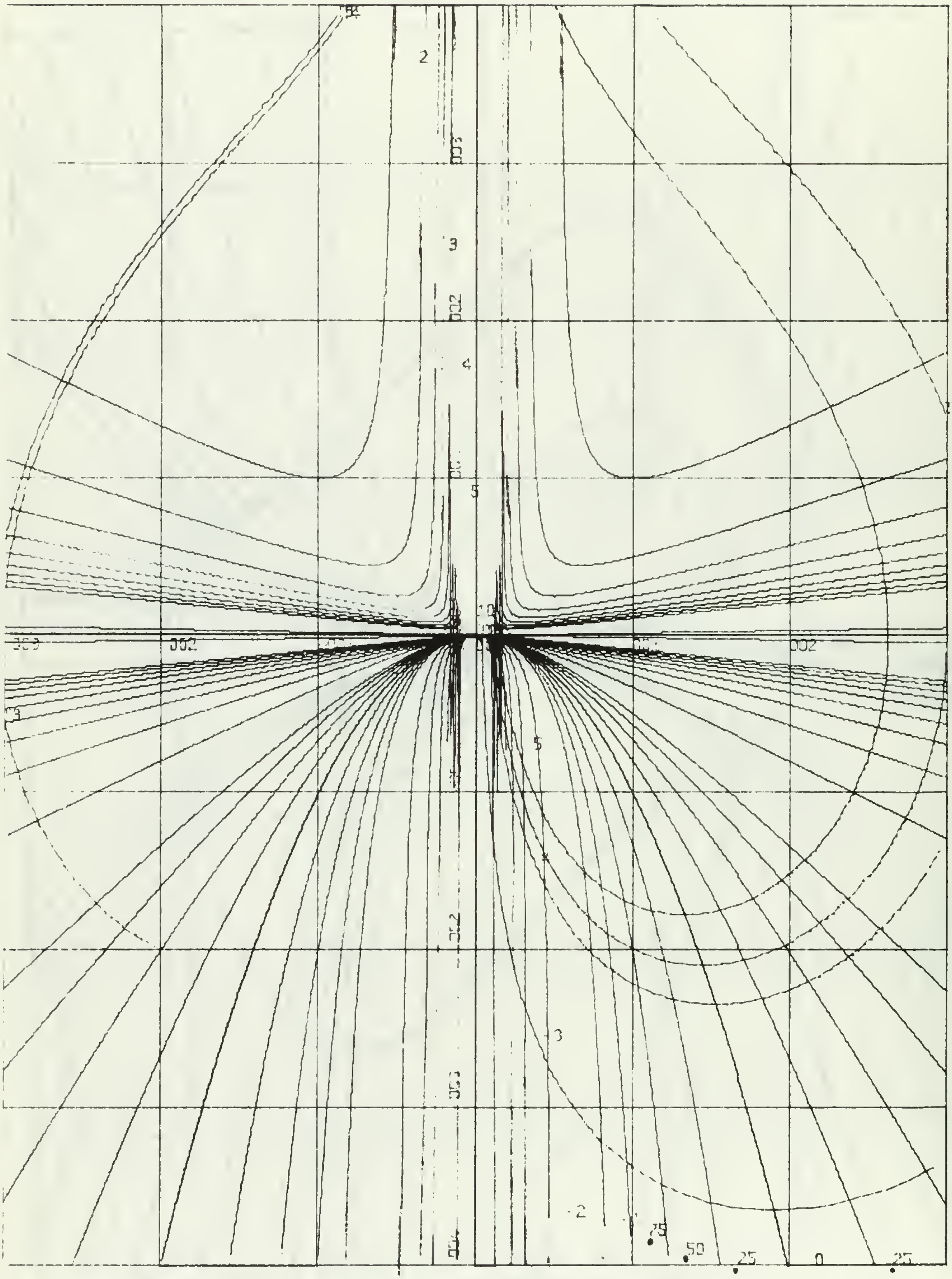


Fig. 4-8. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 9: $\ddot{x} + \dot{x} + \sin x = 0$

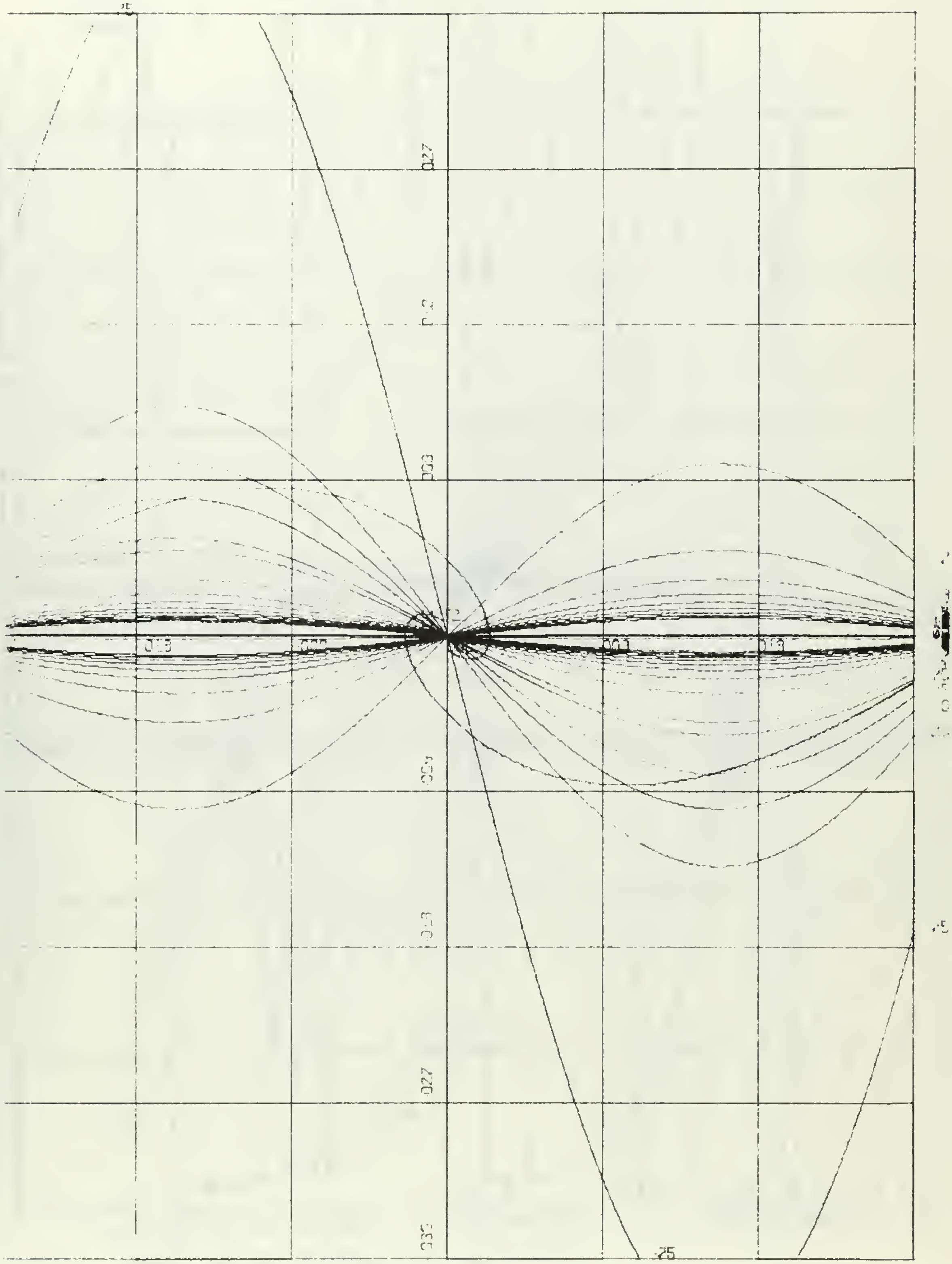


Fig. 4-9. X-Scale = 9.00E-01 Units Inch.
Y-Scale = 9.00E-01 Units Inch.

Example 10: $\ddot{x} + (1-|x|)\dot{x} + x = 0$

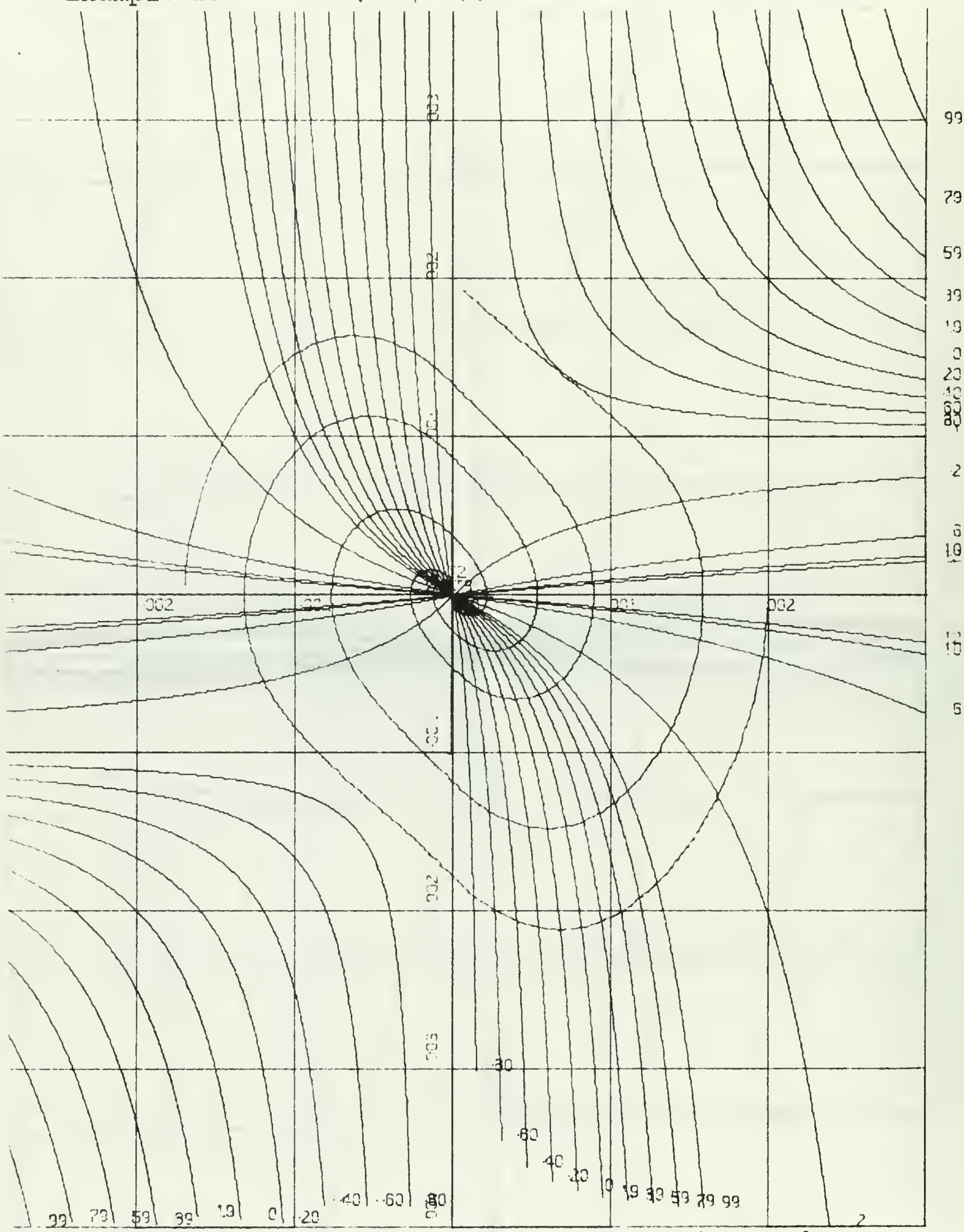


Fig. 4-10.

X-Scale = 1.00E+00 Units Inch.

Y-Scale = 1.00E+00 Units Inch.

Example 11: $\ddot{x} + \frac{g}{\ell} \sin x = 0$

$$\frac{g}{\ell} = 2$$

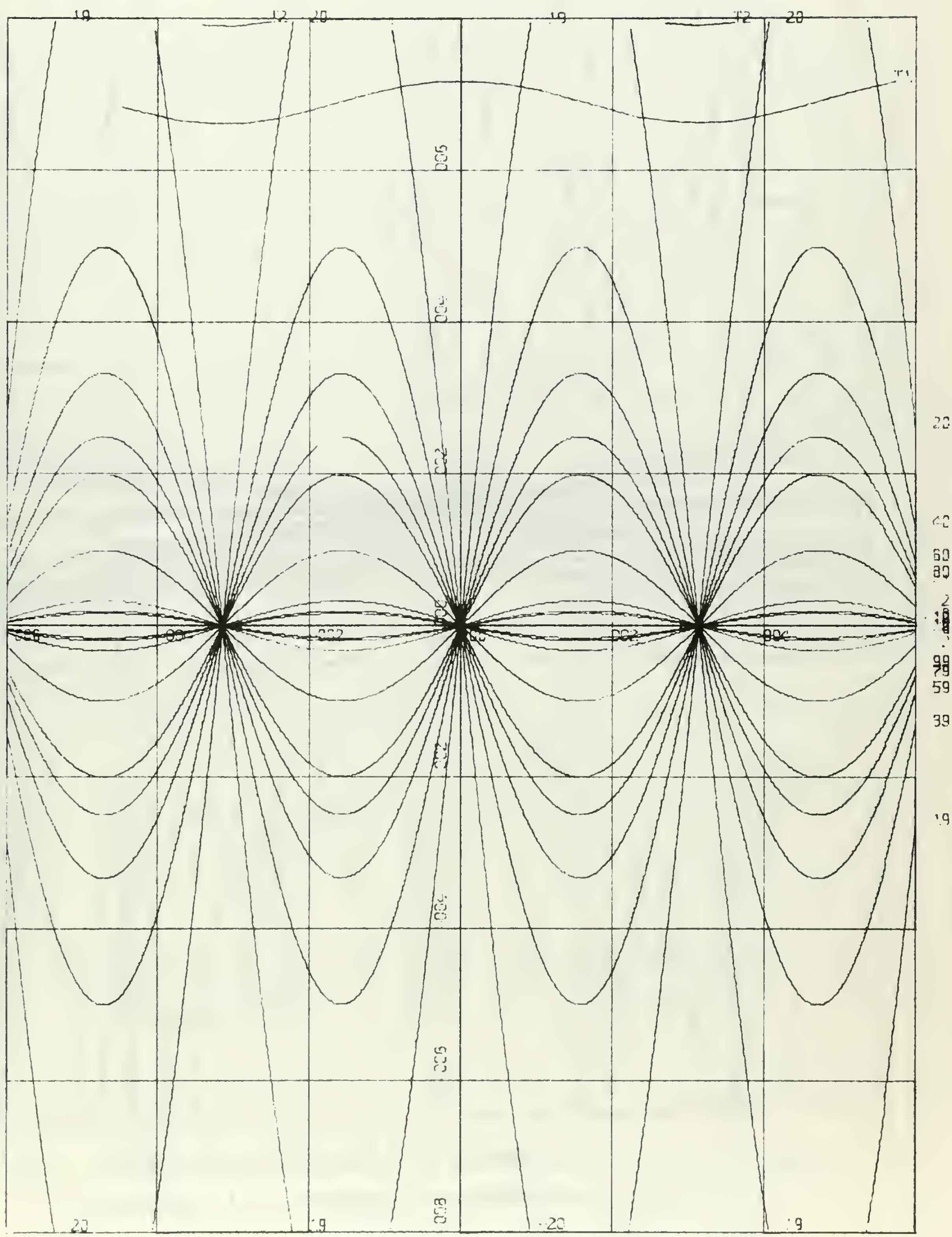


Fig. 4-11. X-Scale = 2.00E+00 Units Inch.
Y-Scale = 2.00E+00 Units Inch.

Example 12: $\ddot{x} + (1-x^2)\dot{x} + x = 0$

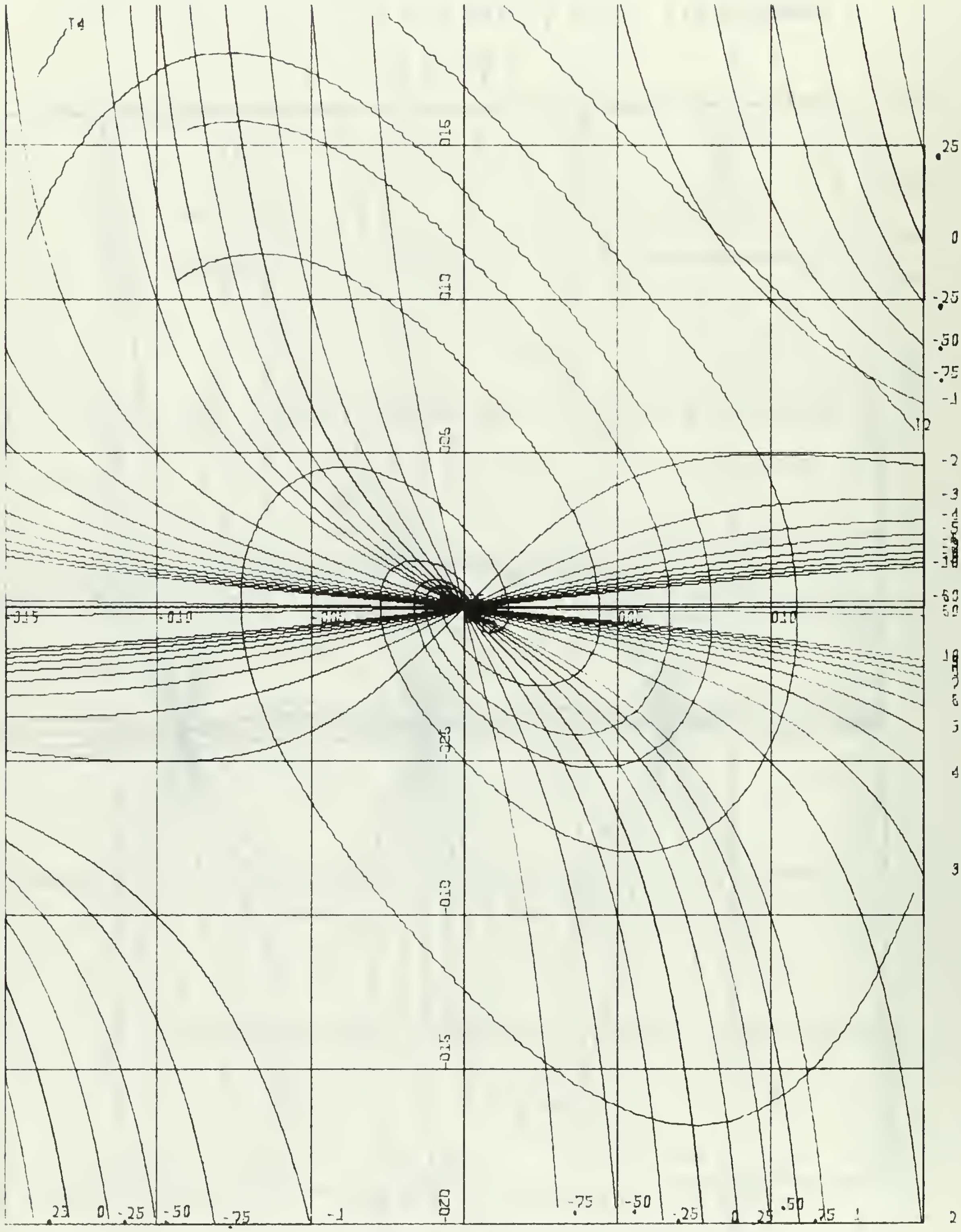


Fig. 4-12. X-Scale = 5.99E-01 Units Inch.
Y-Scale = 5.00E-01 Units Inch.

Example 13: $\ddot{x} + 2x\dot{x} + x = 0$

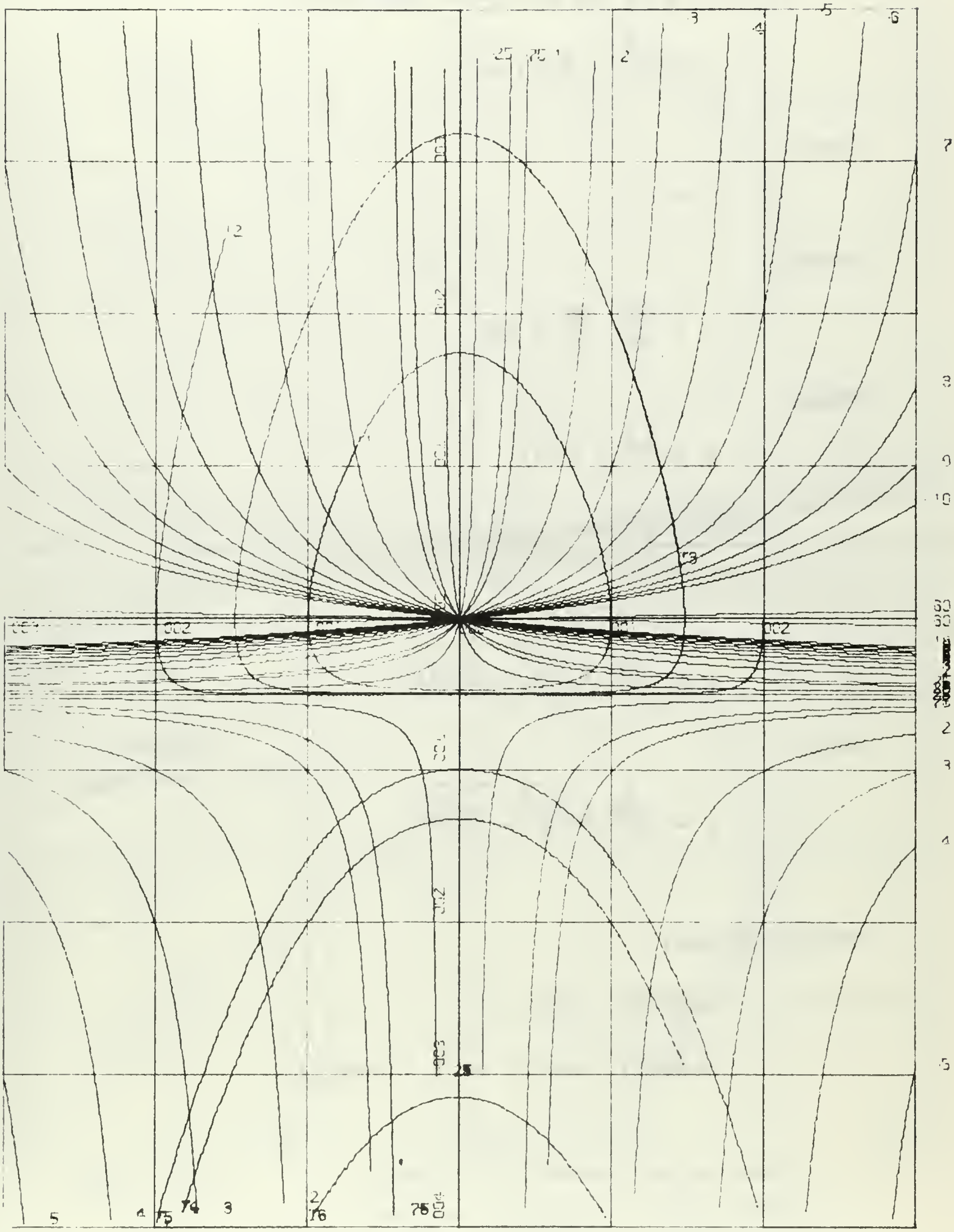


Fig. 4-13. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

B. $\ddot{x} + \dot{x}^2 + G(x) = 0$ TYPE EQUATIONS (SEE PROGRAM 2)

(WHERE $G(x)$ IS ANY FUNCTION OF x)

$$\ddot{x} + \dot{x}^2 + G(x) = 0$$

Let

$$\dot{x} = y \,,$$

then

$$\ddot{x} = \frac{dy}{dx}\,\frac{dx}{dt} = My$$

thus

$$\ddot{x} + \dot{x}^2 + G(x)$$

$$= My + y^2 + G(x) = 0$$

$$y^2 + My + G(x) = 0$$

$$y = \frac{-M + \sqrt{M^2 - 4G(x)}}{2}$$

and

$$y = \frac{-M - \sqrt{M^2 - 4G(x)}}{2}$$

equations of isoclines

Trajectories:

```
ZDØT(1) = Z(2)

ZDØT(2) =-Z(2) ** 2 - G(Z(1)).
```

Example 14: $\ddot{x} + \dot{x}^2 + x = 0$

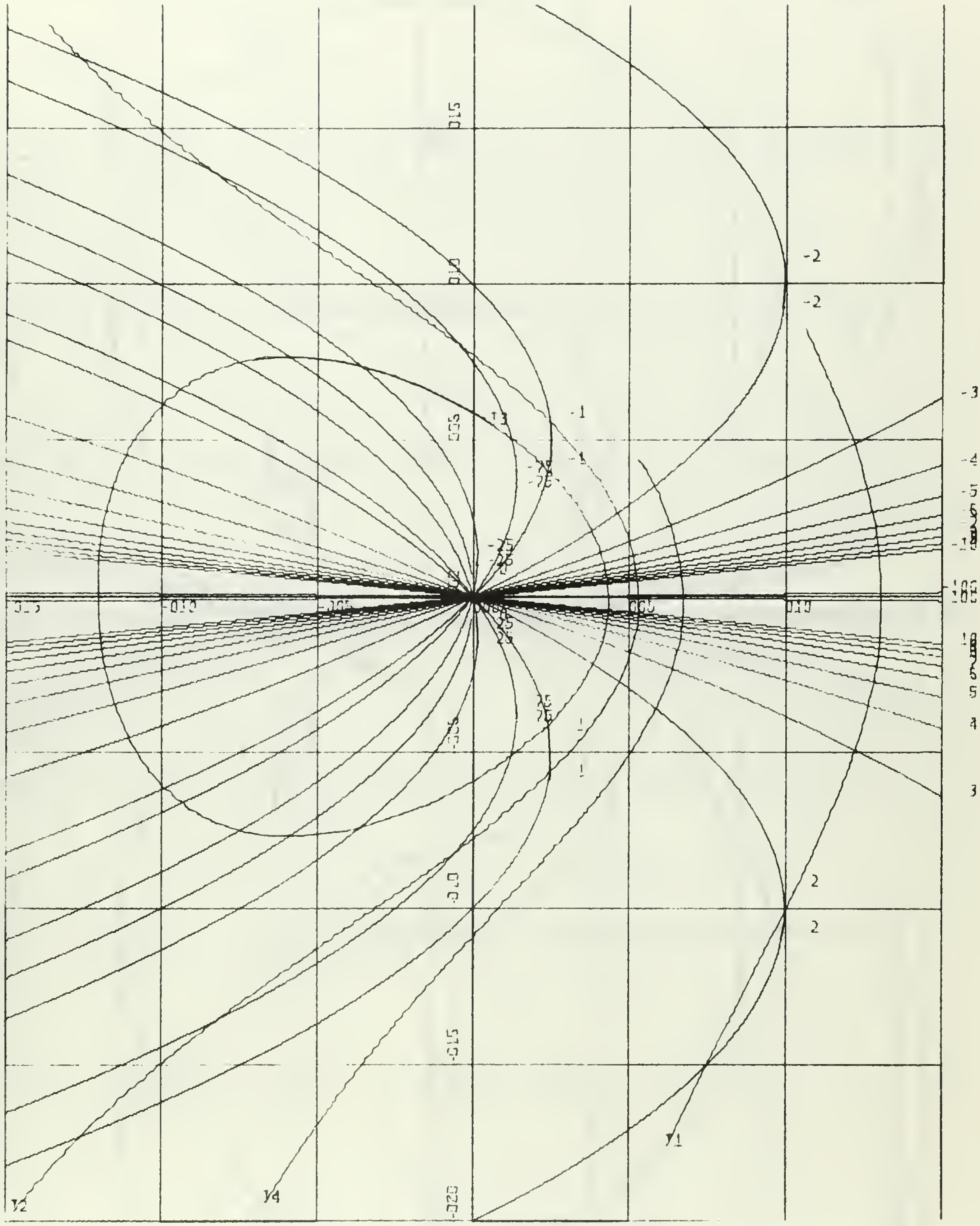


Fig. 4-14. X-Scale = 5.00E-01 Units Inch.

Y-Scale = 5.00E-01 Units Inch.

Example 15: $\ddot{x} + \dot{x}^2 + x^2 = 0$

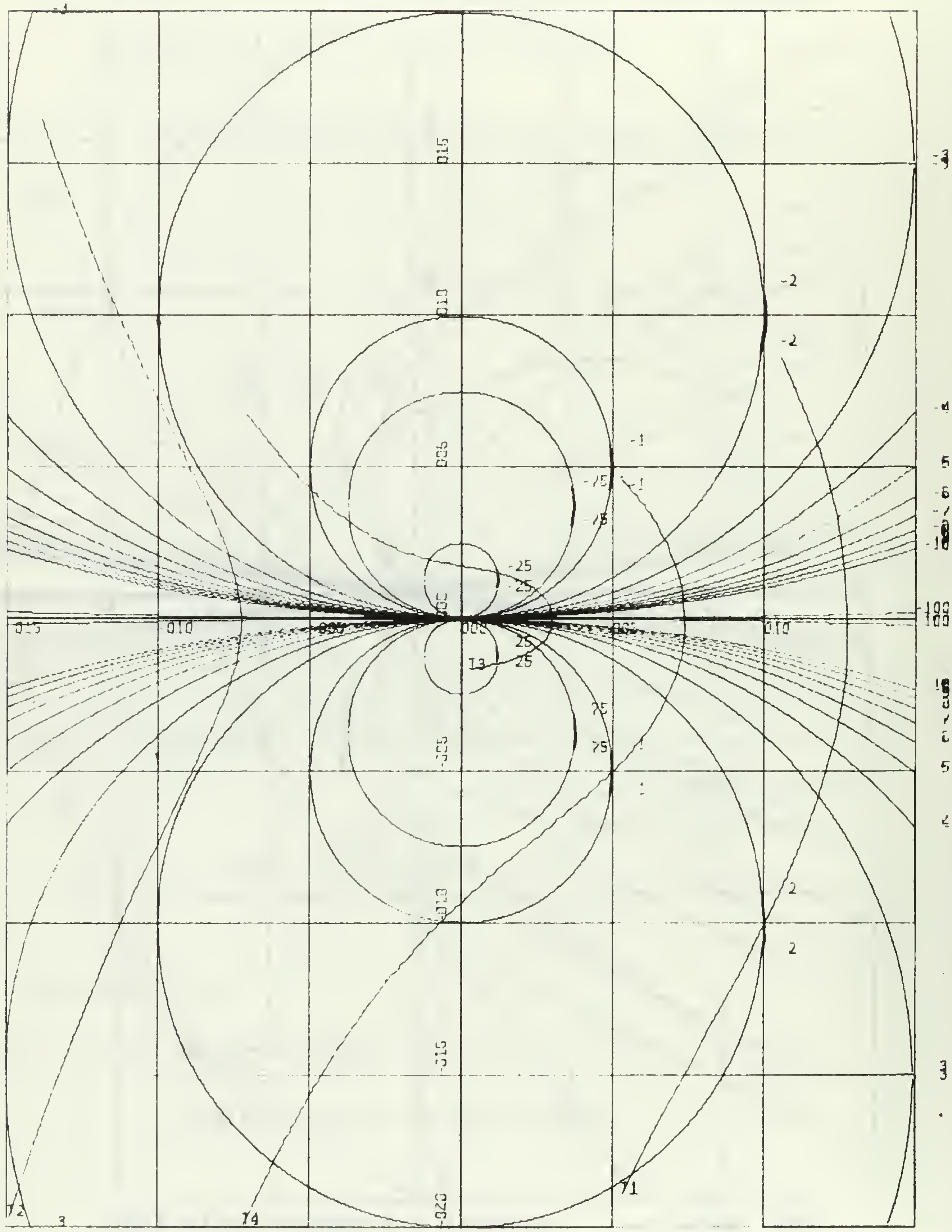


Fig. 4-15. X-Scale = 5.00E-01 Units Inch.
Y-Scale = 5.00E-01 Units Inch.

Example 16: $\ddot{x} + \dot{x}^2 + x|x| = 0$

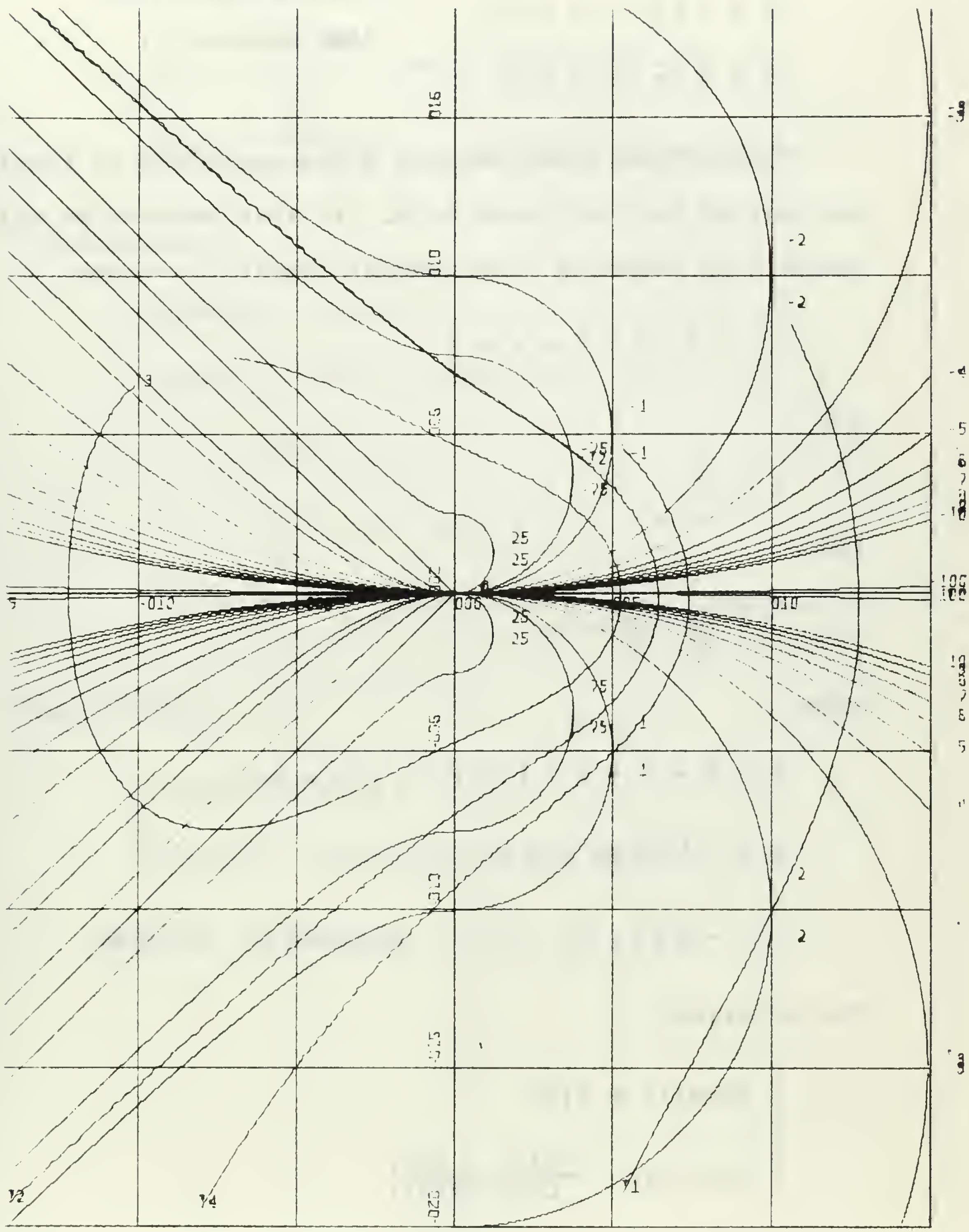


Fig. 16. X-Scale = 5.00E-01 Units Inch.
Y-Scale = 5.00E-01 Units Inch.

C. $|\dot{x}|\ddot{x} + \dot{x} + x = 0$

$\ddot{x} + x|\dot{x}| + x = 0$

$\ddot{x} = \dot{x}|\dot{x}| + x = 0$

TYPE OF EQUATIONS

(SEE PROGRAM 3)

In Section A and Section B the equations of isoclines are solved by y in terms of x. In this section we will solve x in terms of y to prevent imaginary values.

1. $|\dot{x}|\ddot{x} + \dot{x} + x = 0$

Let

$$y = \dot{x} ,$$

then

$$\ddot{x} = \frac{dy}{dx}\frac{dx}{dt} = My$$

thus

$$|\dot{x}|\ddot{x} + \dot{x} + x = |y|My + y + x = 0$$

$$x = -|y|My - y$$

$$= -y(|y|M + 1) \quad \text{equation of isocline.}$$

Trajectories:

$$\begin{cases} ZD\phi T(1) = Z(2) \\ ZD\phi T(2) = \dfrac{-Z(2) - Z(1)}{DABS\ (Z(2))} \end{cases}$$

2. $\ddot{x} + x|\dot{x}| + x = 0$

$\ddot{x} + x|\dot{x}| + x = My + x|y| + x = 0$

$x = -\dfrac{My}{1 + |y|}$ equation of isoclines.

Trajectories:

```
ZDØT(1) = Z(2)

ZDØT(2) =-Z(1)* DABS(Z(2)) - Z(1)
```

3. $\ddot{x} + \dot{x}|\dot{x}| + x = 0$

$\ddot{x} + \dot{x}|\dot{x}| + x = My + y|y| + x = 0$

$x = -My - y|y| = -y(M + |y|)$ equation of isoclines.

Trajectories:

```
ZDØT(1) = Z(2)

ZDØT(2) =-Z(2)* DABS(Z(2)) - Z(1).
```

Example 17: $\ddot{x} + x|\dot{x}| + x = 0$

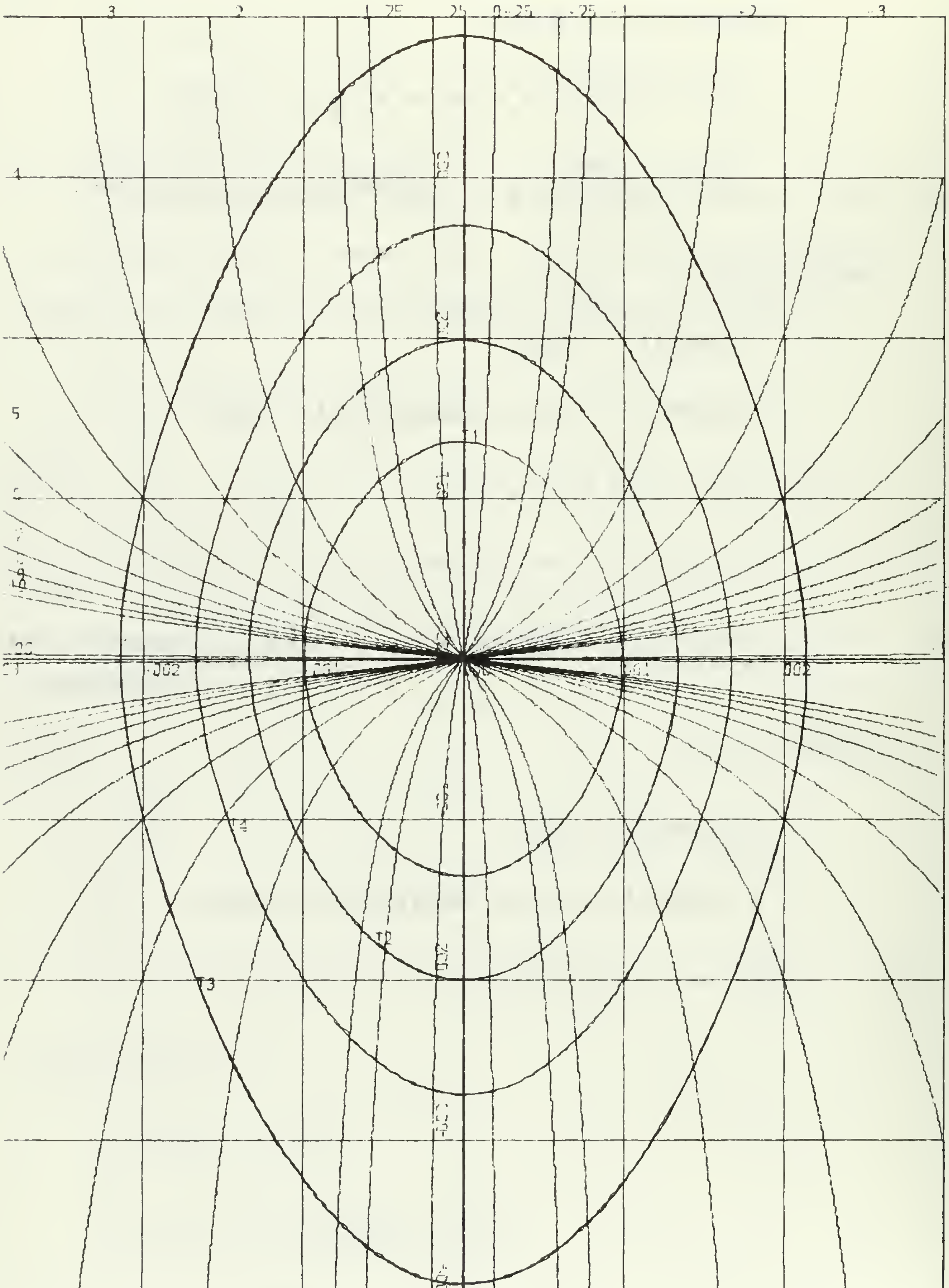


Fig. 4-17. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 18: $\ddot{x} + \dot{x}|\dot{x}| + x = 0$

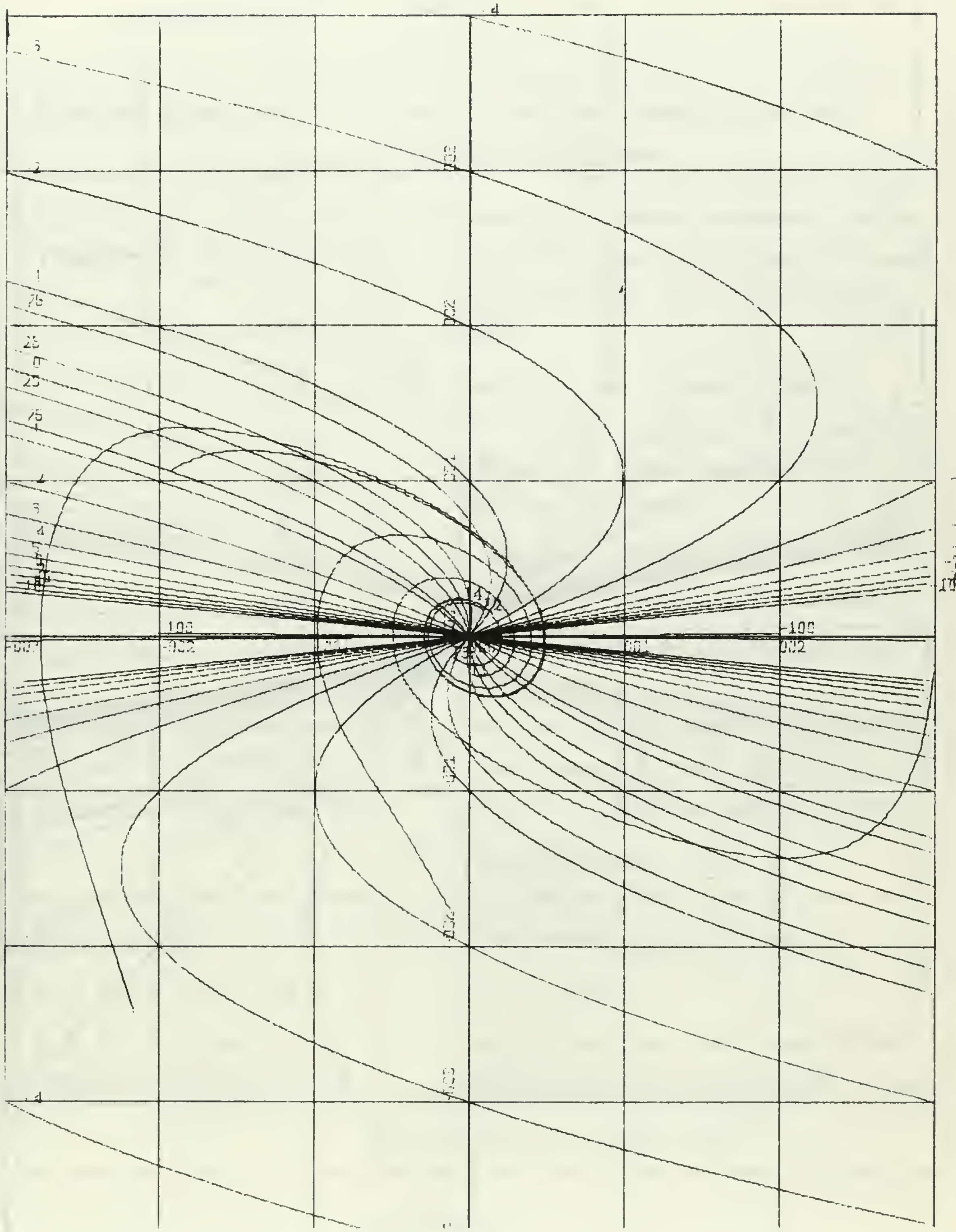


Fig. 4-18. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

Example 19: $|\dot{x}|\ddot{x} + \dot{x} + x = 0$

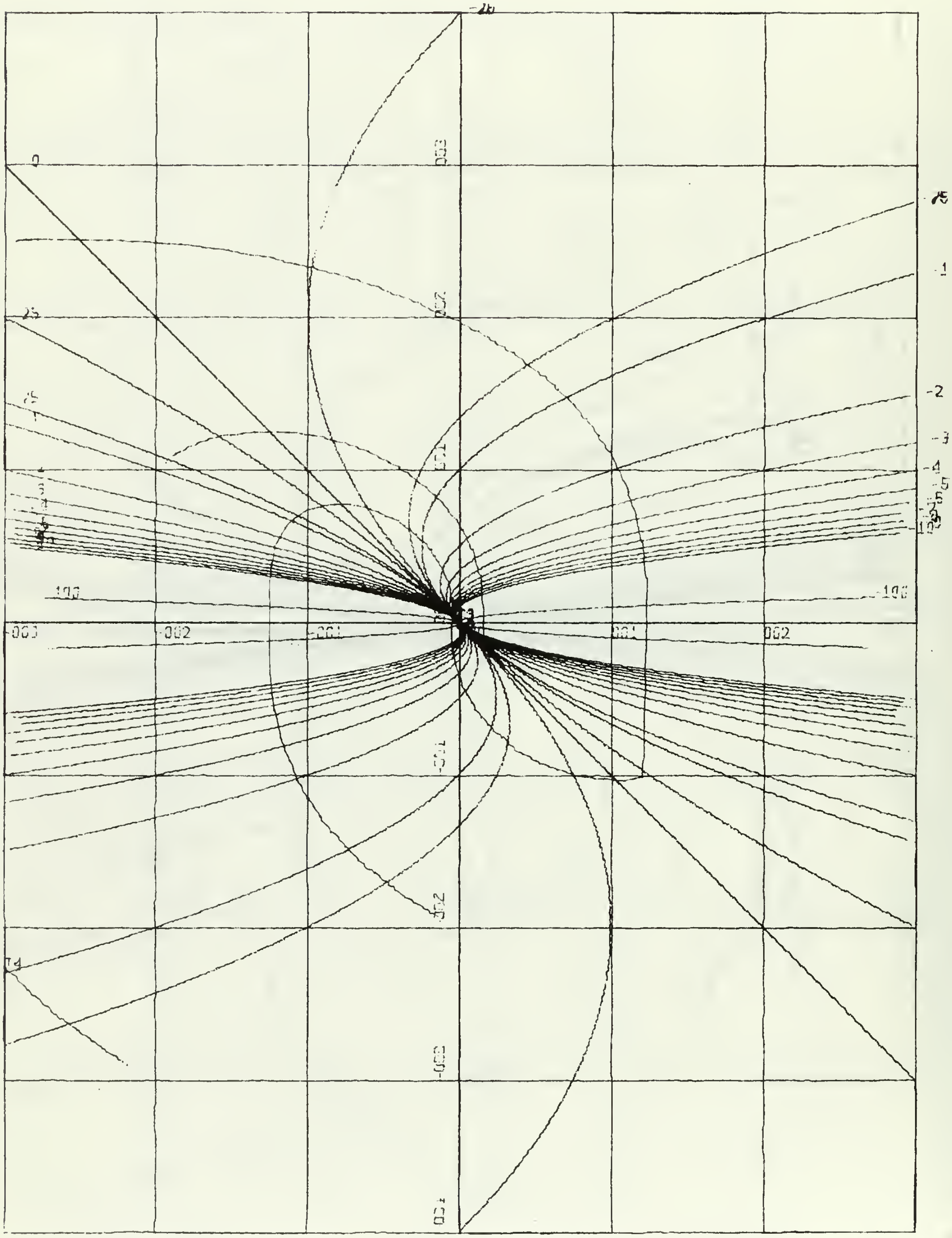


Fig. 4-19. X-Scale = 1.00E+00 Units Inch.
Y-Scale = 1.00E+00 Units Inch.

D. USE OF PROGRAMS

1. Table 1.

Type of Equation	How to Set the Subroutine and Function Programs
$\ddot{x} + F(x)\dot{x} + G(x) = 0$ (Refer to Section A)	(1) Need to compare standard form to give real function F(x) and G(x) in each function subprogram. (2) Rewrite the given equation $\ddot{x} + F(x)\dot{x} + G(x) = 0$ into a system of equations in subroutine program. ZDØT(1) = Z(2) ZDØT(2) =-F(x)*Z(2)- G(x) (F(x), G(x) should change to z form)
$\ddot{x} + \dot{x}^2 + G(x) = 0$ (Refer to Section B)	(1) Need to give real function G(x) in function subprogram. (2) Same as above (2).
$\lvert\dot{x}\rvert\ddot{x} + x = 0$ $\ddot{x} + x\lvert\dot{x}\rvert + x = 0$ $\ddot{x} + \dot{x}\lvert\dot{x}\rvert + x = 0$ (Refer to Section C)	(1) Isoclines: Let $\dot{x} = y$ solve x in terms of y and give this equation in main program. (2) Same as above (2).

Table.1.

b. Input deck, See Table 2.

Sequence of Cards	Purpose	Punch in Column
1st Card	Scale Number of Cards of different ranges of slope	One significant number Column 1-10 Column 11-20
2nd Card	Title	Column 1-48
3rd Card	Title	Column 1-48
4th Card . . . . N No. of cards depends on number of different ranges of slope cards.	Different ranges	Column 1-10 Initial value Column 11-20 Increment Column 21-30 Final value
(N+1)th Card	Time increment Number of trajectories	Column 1-10 Column 11-20
(N+2)th Card . . . K No. of cards depends on number of trajectories.	Initial time Final time Initial x Initial y	Column 1-10 Column 11-20 Column 21-30 Column 31-40
(K+1)th Card	Blank	Blank

Table 2.

C. NOTE: FOR SHIFTING THE CENTER OF COORDINATES. SEE AND COMPARE PROGRAM 10.

Example 20. $\ddot{x} + \frac{4.88}{1.563}\dot{x}\cos x + \sin x = 0$

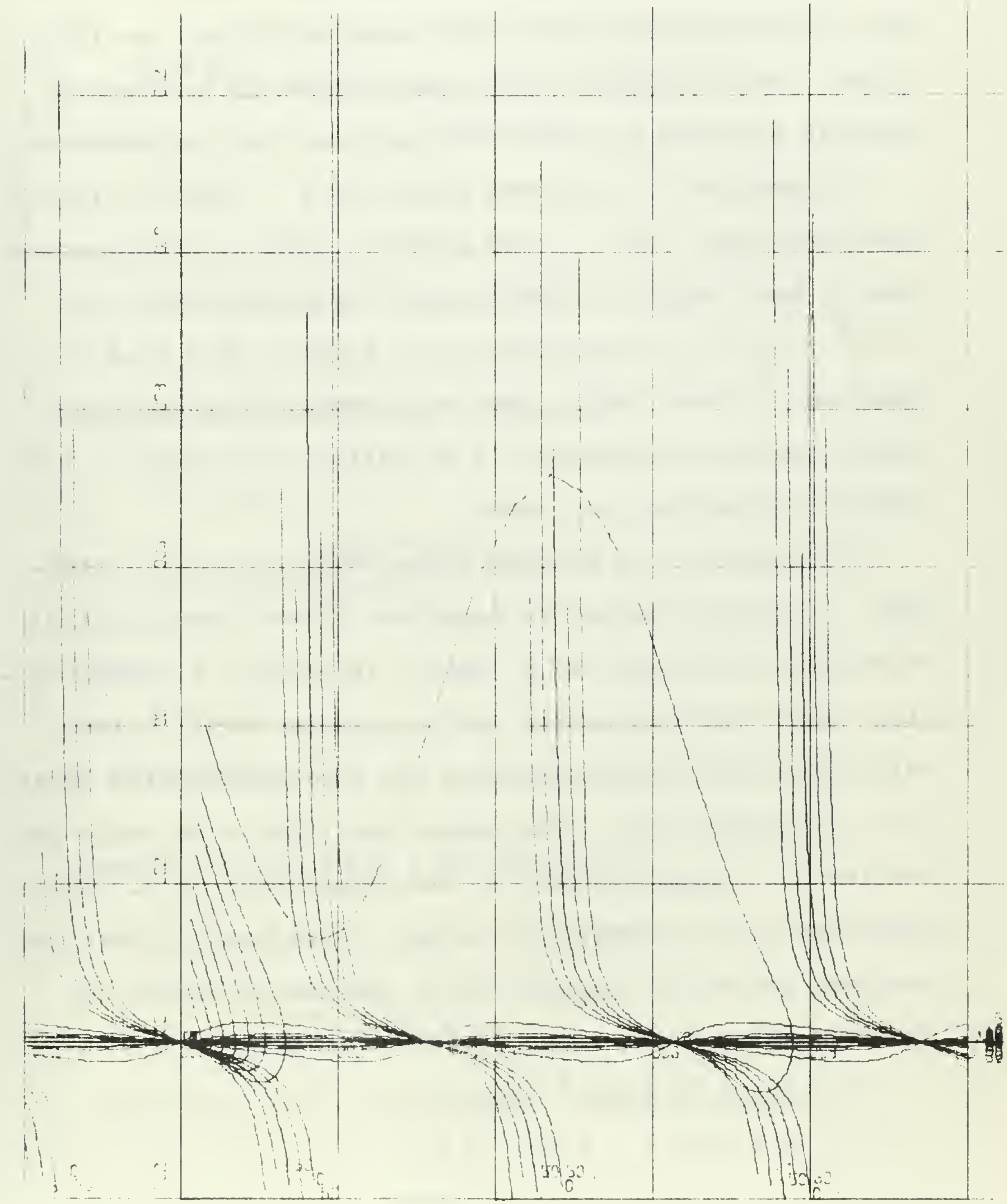


Fig. 4-20. X-Scale = 2.00E+00 Units Inch.
Y-Scale = 2.00E+00 Units Inch.

E. DISCUSSION

In Chapter II it was shown that the isocline method is a very useful general method of phase-plane analysis, the only difficulty being the labor required to get the isoclines. This difficulty has been reduced by developing computer programs to draw both isoclines and trajectories.

In Section A, a program works for $\ddot{x} + F(x)\dot{x} + G(x) = 0$ type equations. This is the general case, and also can be done by hand labor. In Section B, a program works for $\ddot{x} + \dot{x}^2 + G(x) = 0$ type equations. Working this kind of equations by hand labor, some supplementary methods are needed, such as Deekshatulu's or Murthy's, but use of the computer saves time and labor.

In Section C, a program works for some special problems. In this program the equation of isoclines is first solved for x in terms of y (Note: In Section A and B, isocline equations are solved for y in terms of x). After calculating the required points the plotting routine treats y as a function of x. The reason for this is to avoid the problem of complex values in the computation of y. This procedure is not always successful; there are at least two problems for which complex values apparently cannot be avoided. They are:

1. $\ddot{x} + \dot{x}|\dot{x}| + x^2 = 0$
2. $\ddot{x} + \dot{x}|\dot{x}| + x|x| = 0$

For example: Equation $\ddot{x} + \dot{x}\,|\dot{x}\,| + x^2 = 0$

Let

$$\dot{x} = y$$

then

$$\ddot{x} = My$$

thus

$$\ddot{x} + \dot{x}\,|\dot{x}\,| + x^2 = My + y\,|y\,| + x^2 = 0$$

<u>Case a</u>: Solving for y in terms of x

$$y^2 + My + x^2 = 0$$

$$y = \frac{-M \pm \sqrt{M^2 - 4x^2}}{2}$$

when

$$M < 4x^2$$

y is complex.

<u>Case b</u>: Solving for x in terms of y

$$x^2 = -[y\,|y\,| + My]$$

$$x = \sqrt{-[y\,|y\,| + My]}$$

when $[y\,|y\,| + My]$ is positive x becomes complex.

V. GENERATING PROGRAMS TO DRAW ISOCLINES AND TRAJECTORIES WITH DIVIDING LINES OF THE SECOND-ORDER DIFFERENTIAL EQUATIONS

A. SATURATION

1. Program for Vertical Dividing Lines only

(See Program 4)

Refer to Chapter III, Section B-1.

Example 1. Trajectories: $\ddot{x} + 0.2\dot{x} + x = 0$

(for Inner regions)

Dividing lines

$$x = \pm .3$$

Isoclines:

(See Fig. 5-1)

Inner linear region:

$$\ddot{x} + 0.2\dot{x} + x = 0$$

$$My + 0.2y + x = 0$$

$$y = \frac{-x}{M + 0.2}$$

Saturation regions:

$$\ddot{x} + 0.2\dot{x} \pm 0.3 = 0$$

$$My + 0.2y \pm 0.3 = 0$$

$$y = \frac{\mp 0.3}{M + 0.2}$$

Trajectories:

Inner region

ZDØT(1) = Z(2)

ZDØT(2) = -0.2* Z(2) -Z(1)

Saturation region

ZDØT(1) = Z(2)

ZDØT(2) = -0.2* Z(2) ± 0.3

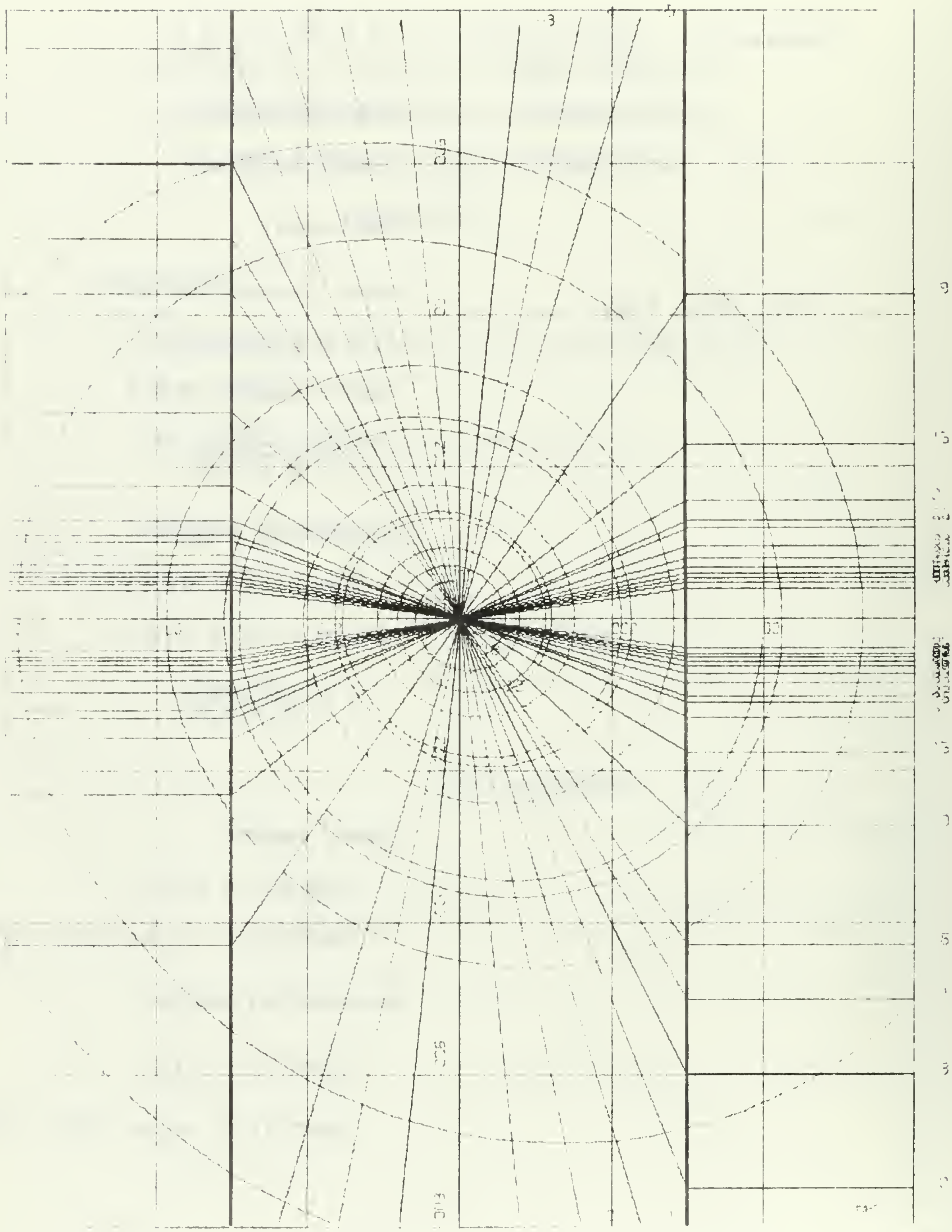


Fig. 5-1. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 2: $\ddot{x} + 0.2\dot{x} + x = 0$

Dividing lines $x = \pm 0.2$

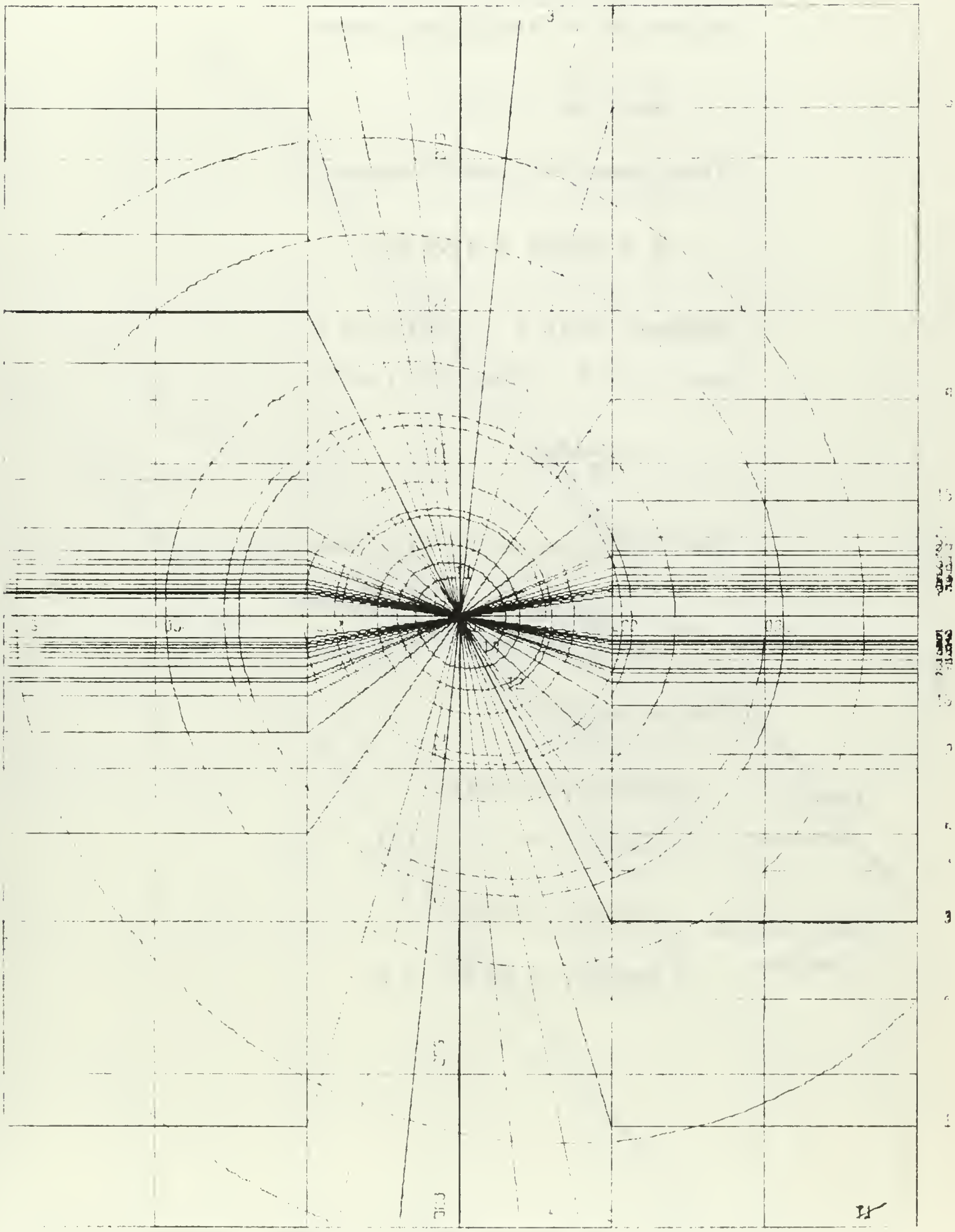


Fig. 5-2. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

2. <u>Program for Vertical Dividing Lines which can be Rotated. (See Program 5)</u>

Equation of dividing lines:

$$Ax + By = \pm C$$

Isoclines in linear region:

$$\ddot{x} + F(x)\dot{x} + G(x) = 0$$

assume $F(x) = 1, \quad G(x) = x$

Let $y = \dot{x}$, then $\ddot{x} = My$

$$y = \frac{\pm x}{M + 1}$$

Isoclines in saturation regions:

$$y = \frac{\pm C}{M + 1}$$

Trajectories:

Inner region
$$\begin{cases} ZD\emptyset T(1) = Z(2) \\ ZD\emptyset T(2) = -1 * Z(2) - Z(1) \end{cases}$$

Saturation region
$$\begin{cases} ZD\emptyset T(1) = Z(2) \\ ZD\emptyset T(2) = -Z(2) \pm C \end{cases}$$

Example 3: Dividing lines: $x + 0.8y = \pm 0.3$

Trajectories : $\ddot{x} + 0.2\dot{x} + x = 0$

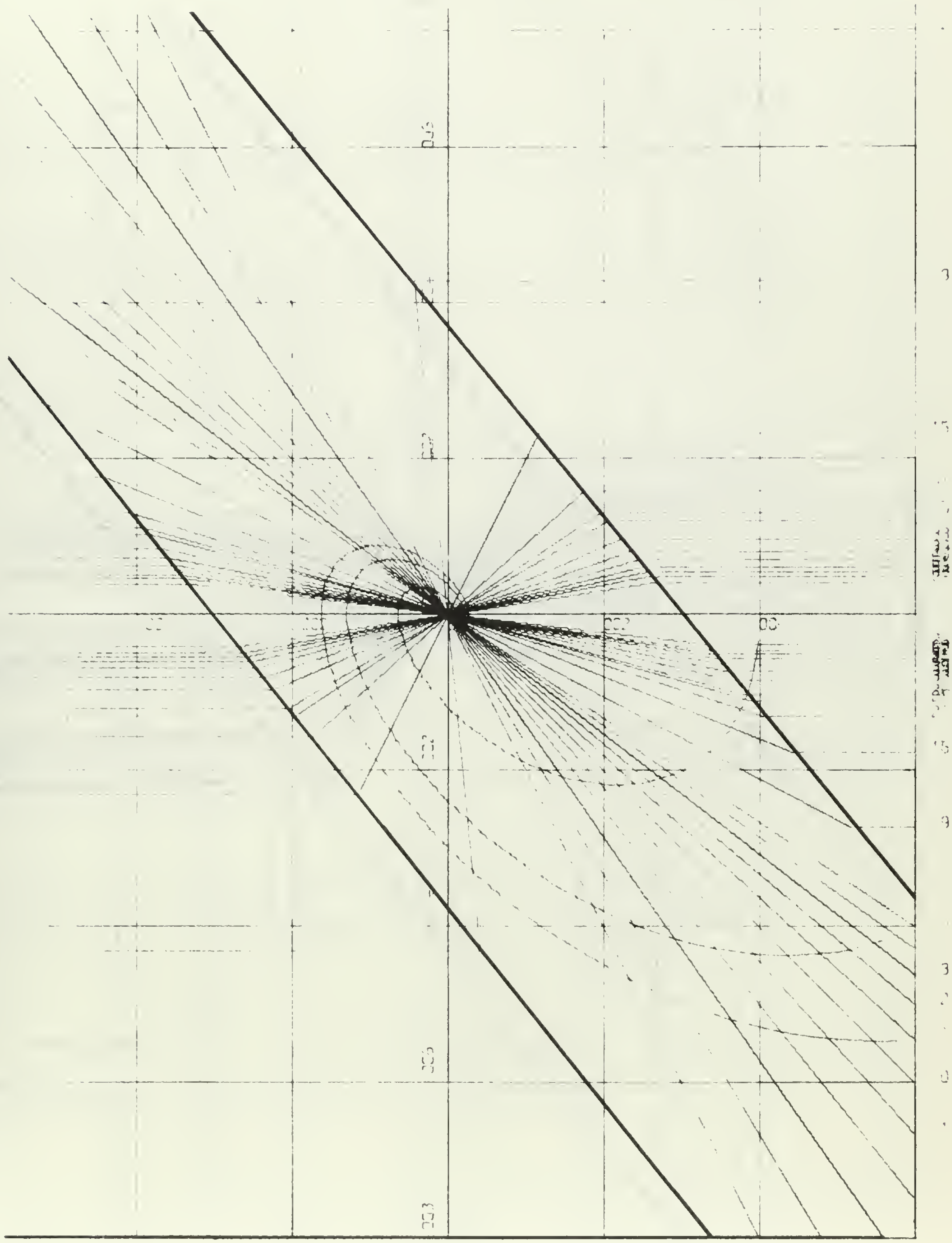


Fig. 5-3. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 4: Dividing lines: $x - 0.8y = \pm 0.3$

Trajectories : $\ddot{x} + 0.2\dot{x} + x = 0$

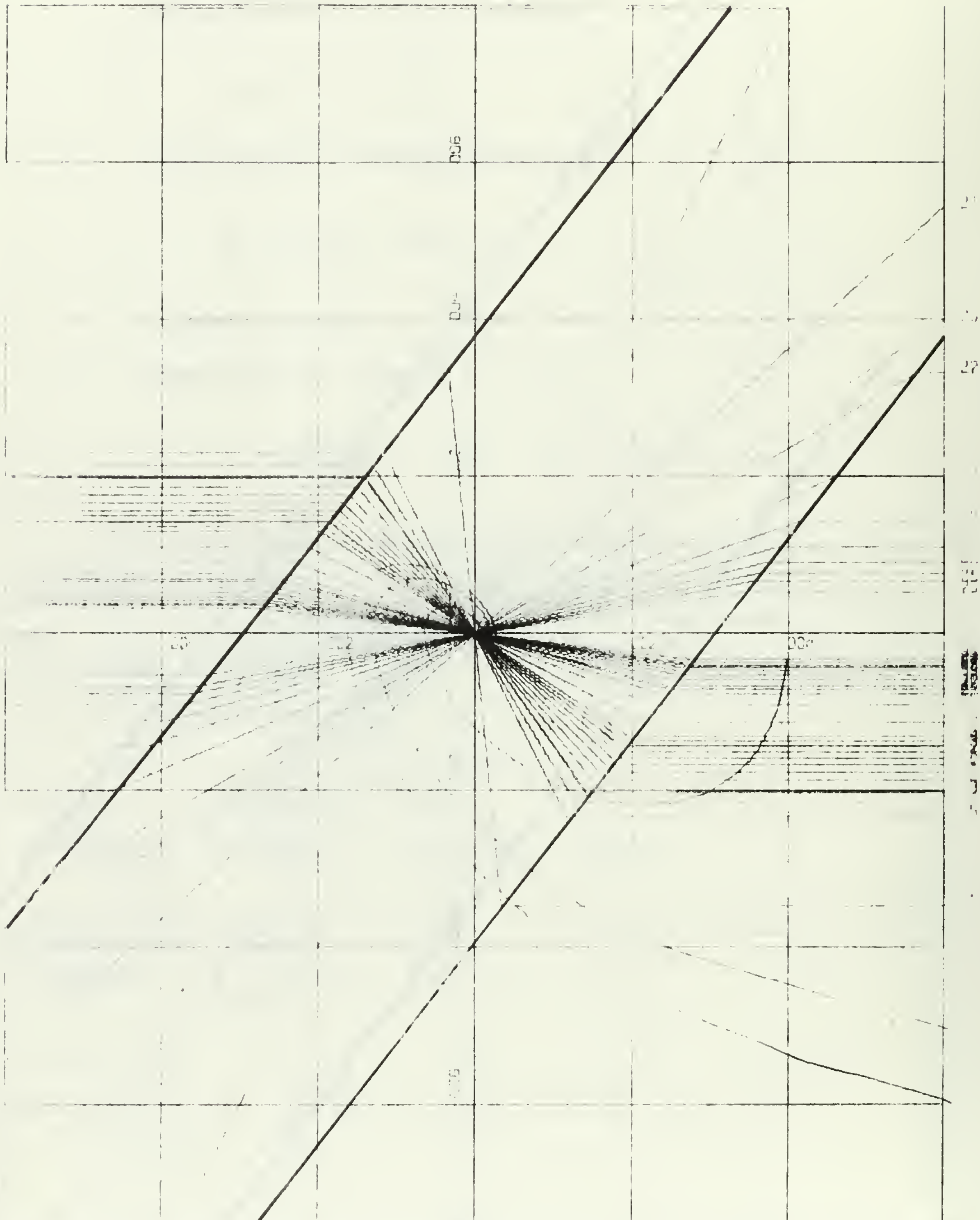
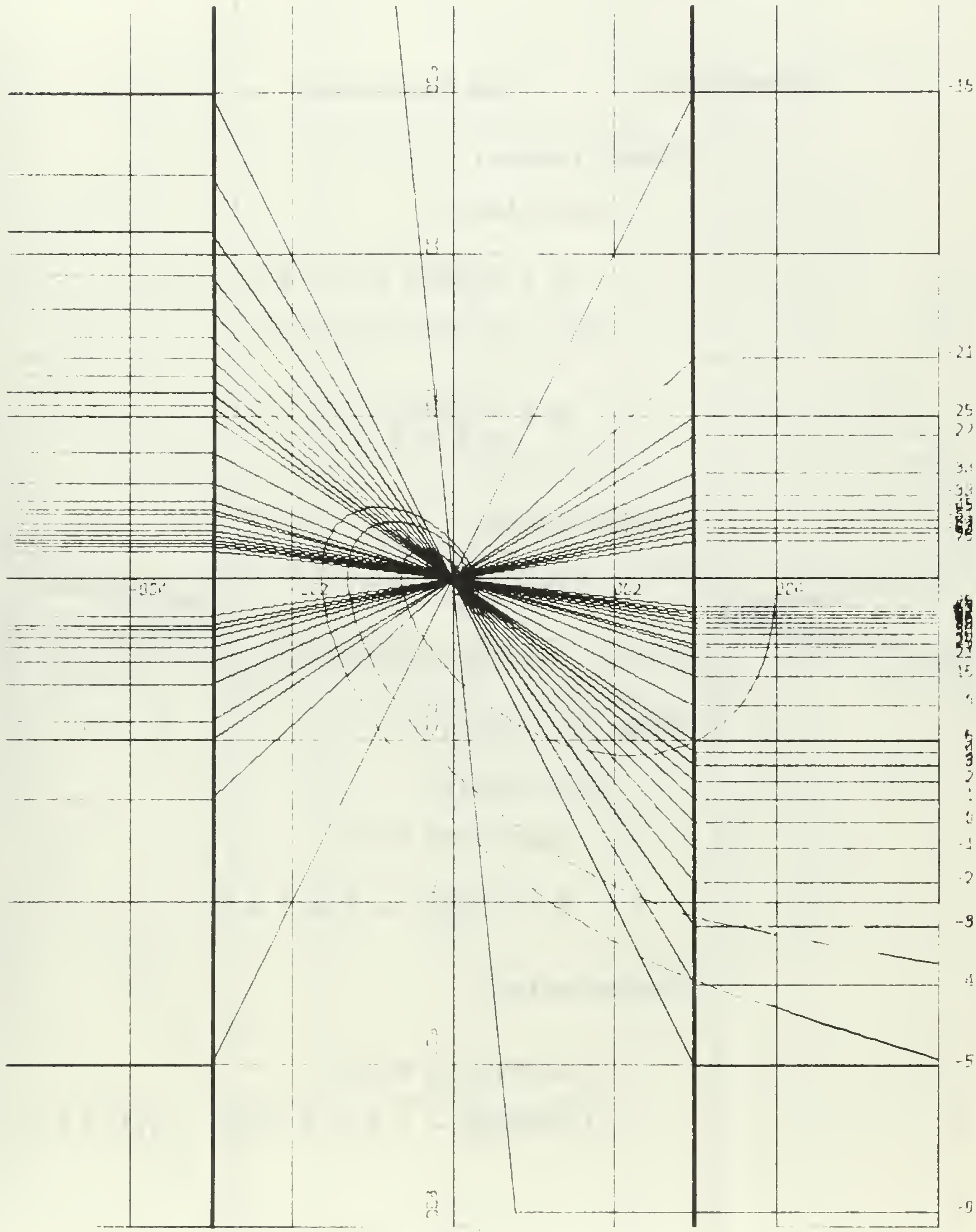


Fig. 5-4. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5: Dividing lines: $x = \pm 0.3$

$(y = 0)$

Trajectories : $\ddot{x} + 0.2\dot{x} + x = 0$



X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Fig. 5-5.

B. DEAD ZONE. (SEE PROGRAM 6)

(Refer to Chapter III Section B-2.)

Example 6: (See Fig. 5-6)

Inner region:

Isoclines:

$$\ddot{x} + 0.2\dot{x} + 0.3 = 0$$

$$My + 0.2y + 0.3 = 0$$

$$M = \frac{-0.3}{M + 0.2}$$

Trajectories:

$$\begin{cases} ZD\phi T(1) = Z(2) \\ ZD\phi T(2) = -0.2 * Z(2) \end{cases}$$

External region:

Isoclines:

Right and left

$$\ddot{x} + 0.2\dot{x} + x \mp 0.3 = 0$$

Trajectories:

$$\begin{cases} ZD\phi T(1) = Z(2) \\ ZD\phi T(2) = - 0.2 * Z(2) - Z(1) \pm 0.3. \end{cases}$$

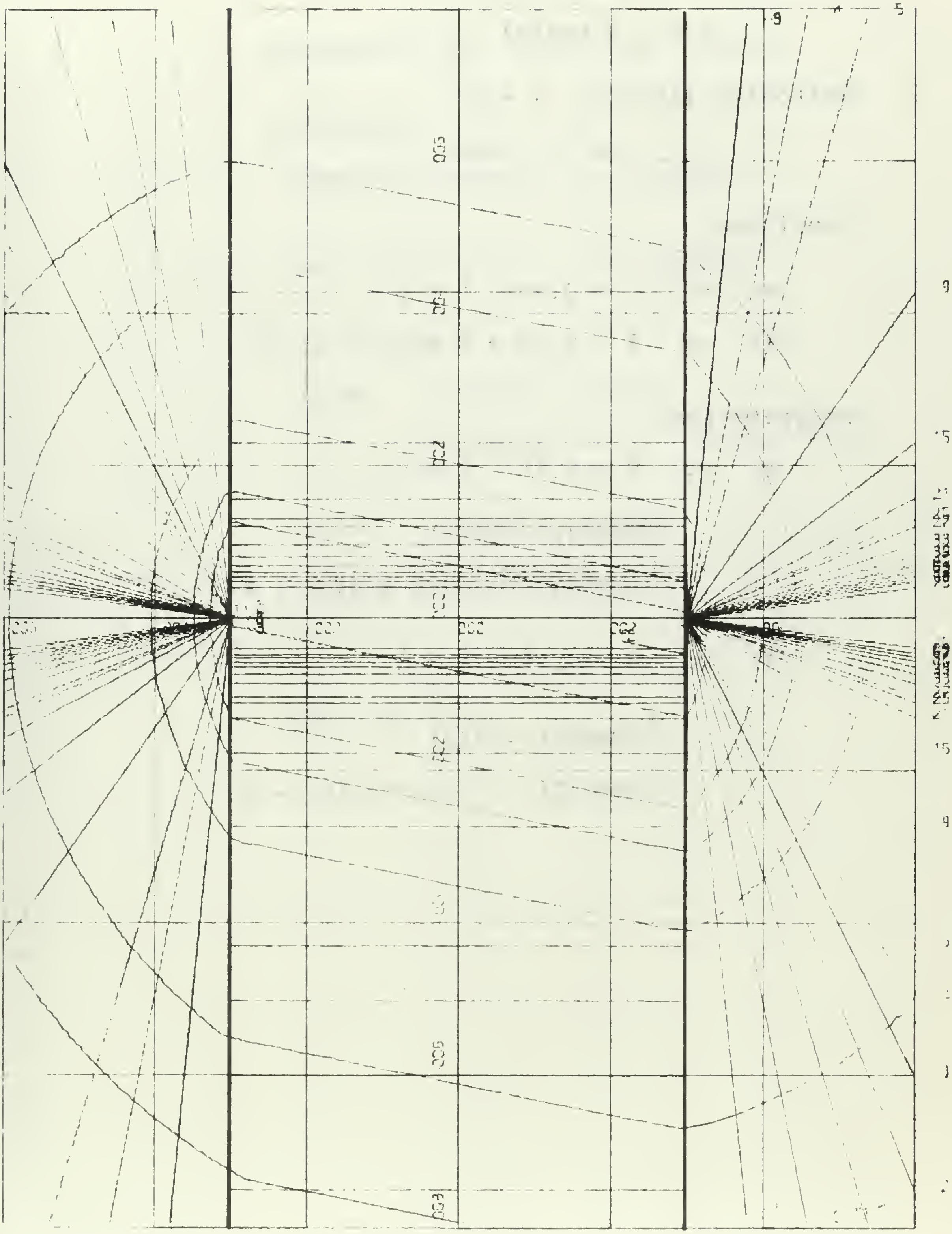


Fig. 5-6. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

C. IDEAL RELAY. (SEE PROGRAM 7, 8, 8A)

For $\ddot{x} + 0.2\dot{x} + u = 0$

$$u = - K \, \mathrm{Sgn}(x)$$

Switching line is $x = 0$

or y axis.

Isoclines:

At $+x$ $\quad \ddot{x} + 0.2\dot{x} - K = 0$

At $-x$ $\quad \ddot{x} + 0.2\dot{x} + K = 0$

Trajectories:

At $+x$ $\quad \ddot{x} + 0.2\dot{x} - K = 0$

$$\begin{cases} \mathrm{ZD\phi T(1)} = \mathrm{Z(2)} \\ \mathrm{ZD\phi T(2)} = - 0.2 * \mathrm{Z(2)} + K \end{cases}$$

At $-x$ $\quad \ddot{x} + 0.2\dot{x} + K = 0$

$$\begin{cases} \mathrm{ZD\phi T(1)} = \mathrm{Z(2)} \\ \mathrm{ZD\phi T(2)} = - 0.2 * \mathrm{Z(2)} - K \end{cases}$$

Example 7: $\ddot{x} + 0.2\dot{x} + u = 0, \quad u = 0.3 \text{ Sgn } x$

Switching line: $x = 0$ or y-axis.

Isoclines:

Right of y-axis: $y = \dfrac{-0.3}{M + 0.2}$

Left of y-axis: $y = \dfrac{0.3}{M + 0.2}$

Trajectories:

Right $\begin{cases} \text{ZD}\phi\text{T}(1) = \text{Z}(2) \\ \text{ZD}\phi\text{T}(2) = -\ 0.2 * \text{Z}(2) + 0.3 \end{cases}$

Left $\begin{cases} \text{ZD}\phi\text{T}(1) = \text{Z}(2) \\ \text{ZD}\phi\text{T}(2) = -\ 0.2 * \text{Z}(2) - 0.3 \end{cases}$

(See Fig. 5-7)

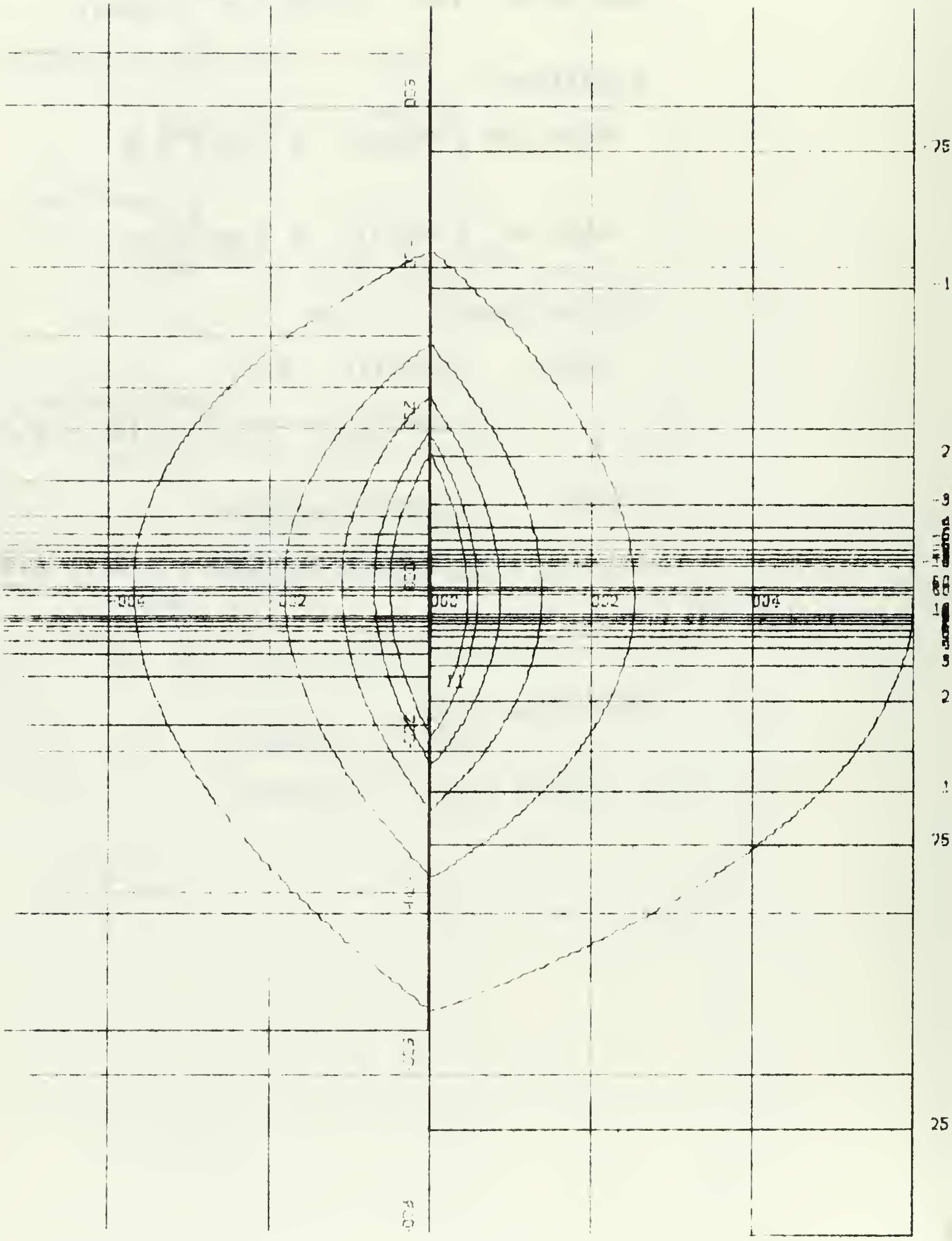


Fig. 5-7. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 8(a): $\ddot{x} + 0.02\dot{x} + u = 0$

$$u = -0.3\,\mathrm{Sgn}(x)$$

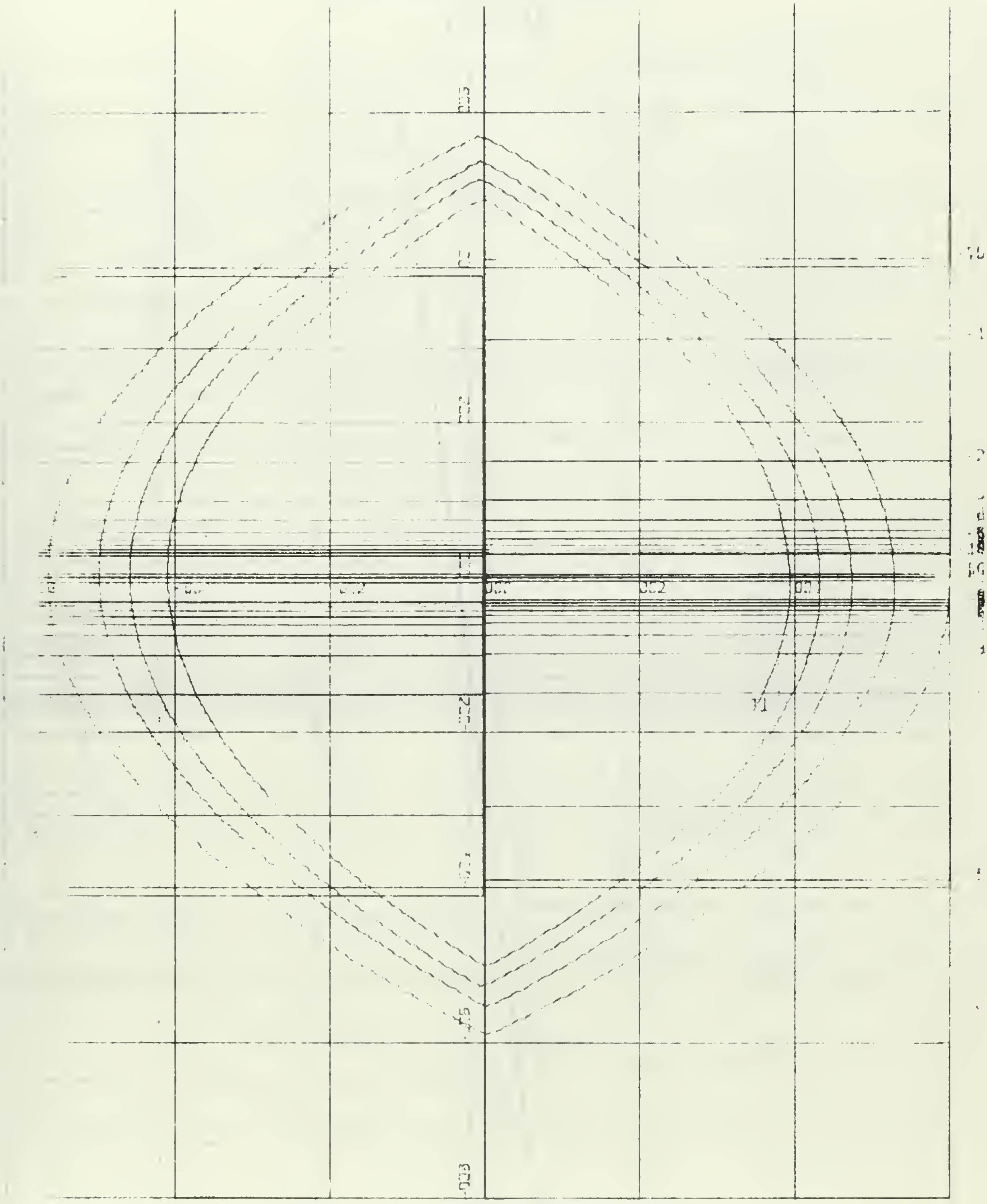


Fig. 5-8(a). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 8(b): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = - 0.3 \operatorname{Sgn}(8x-y)$

$(\dot{x} = y)$

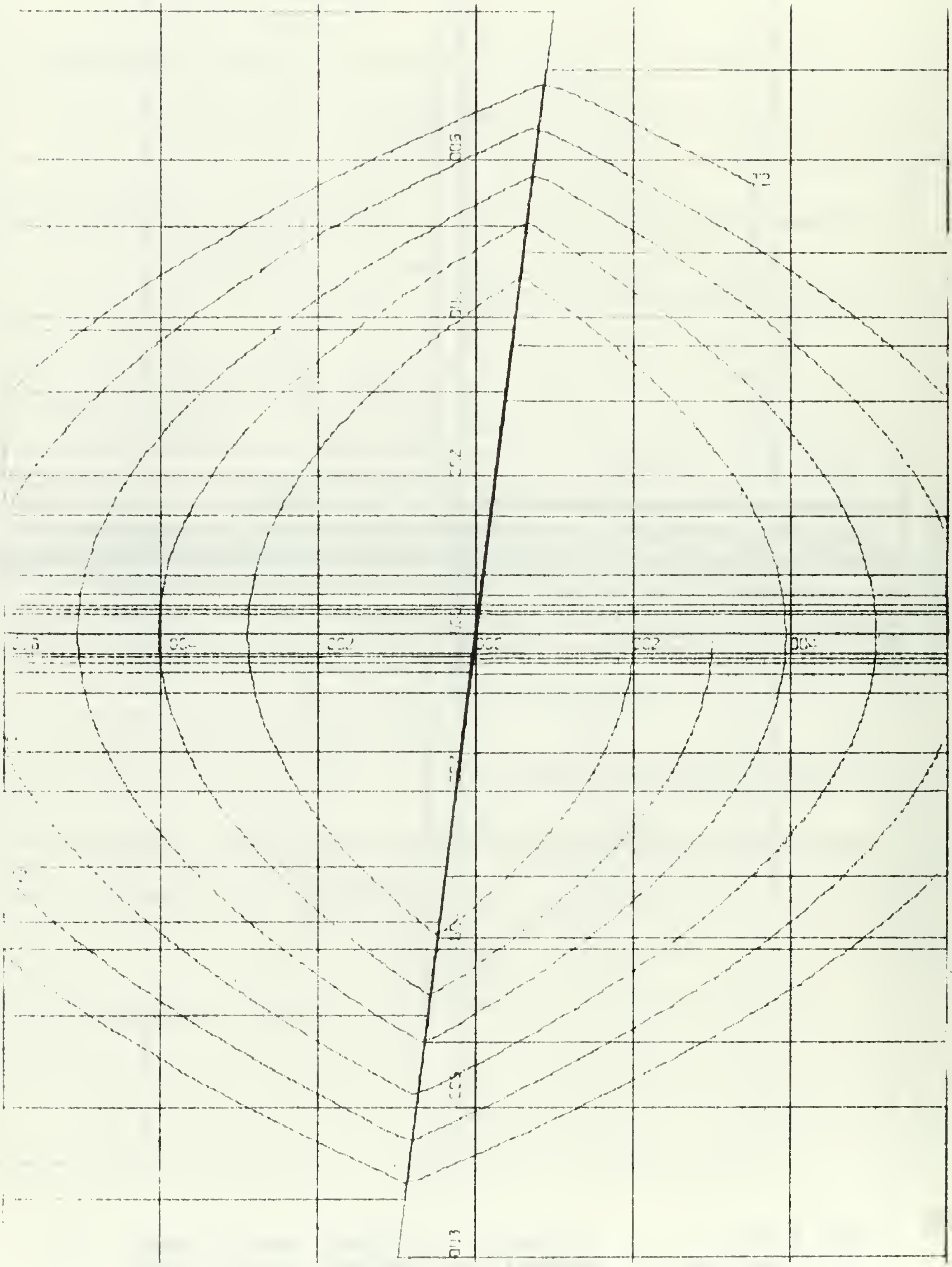


Fig. 5-8(b). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 8(c): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\ \mathrm{Sgn}(8x+y)$

$(\dot{x} = y)$

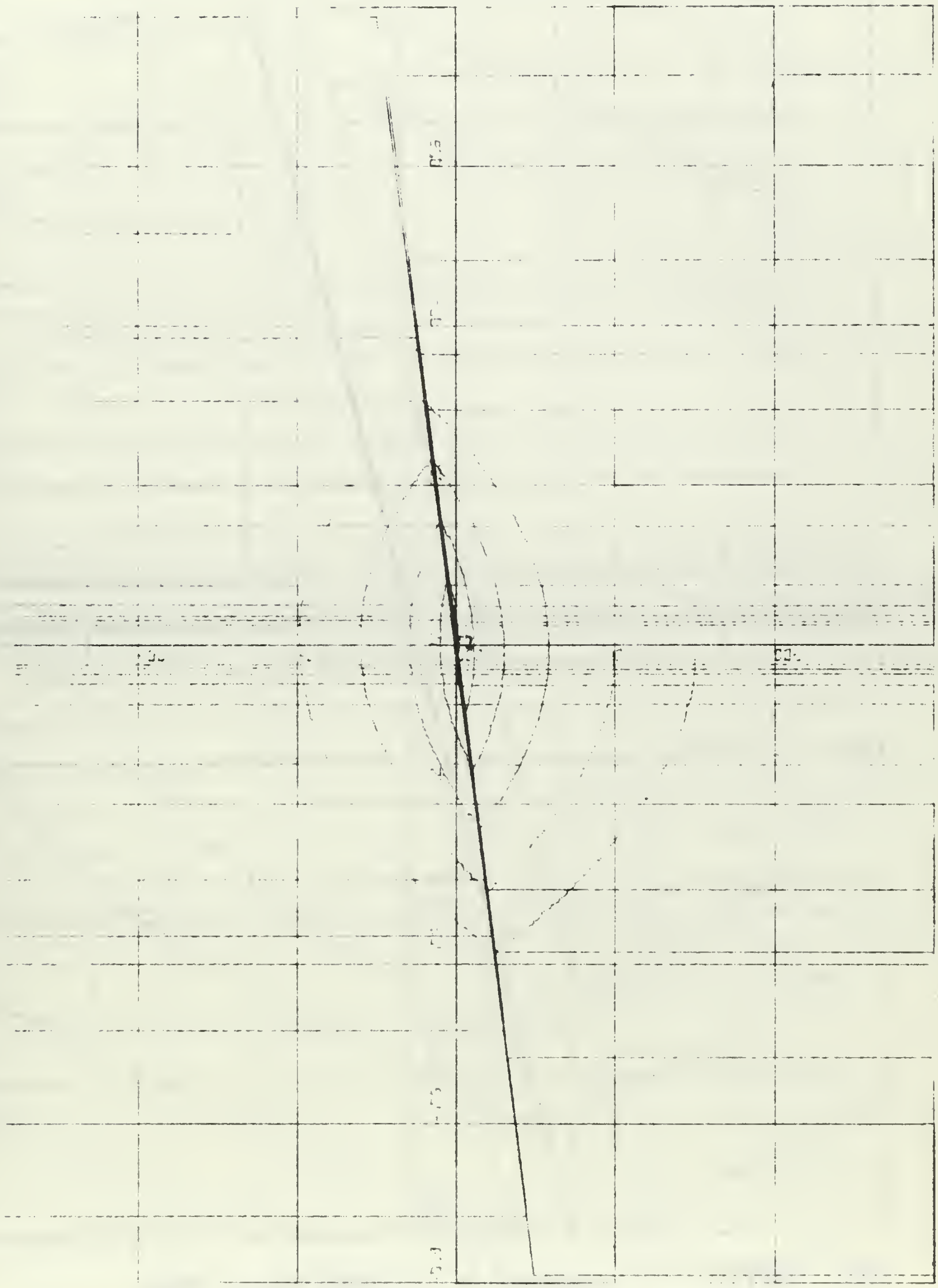


Fig. 5-8(c).

X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(d): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\ \mathrm{Sgn}(4x-y),\ \dot{x} = y$

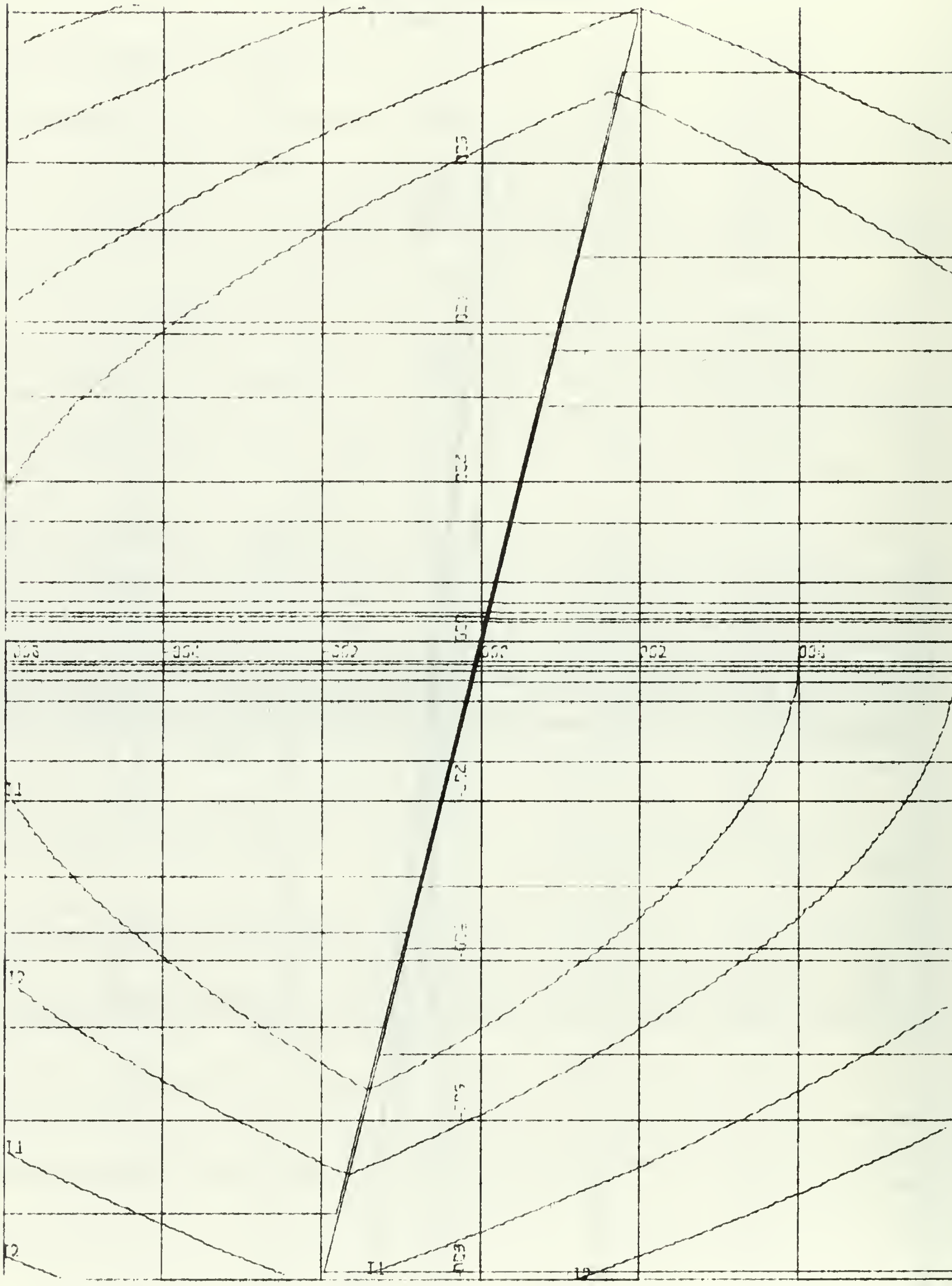


Fig. 5-8(d). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(e): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3 \text{ Sgn } (4x+y), \; \dot{x} = y$

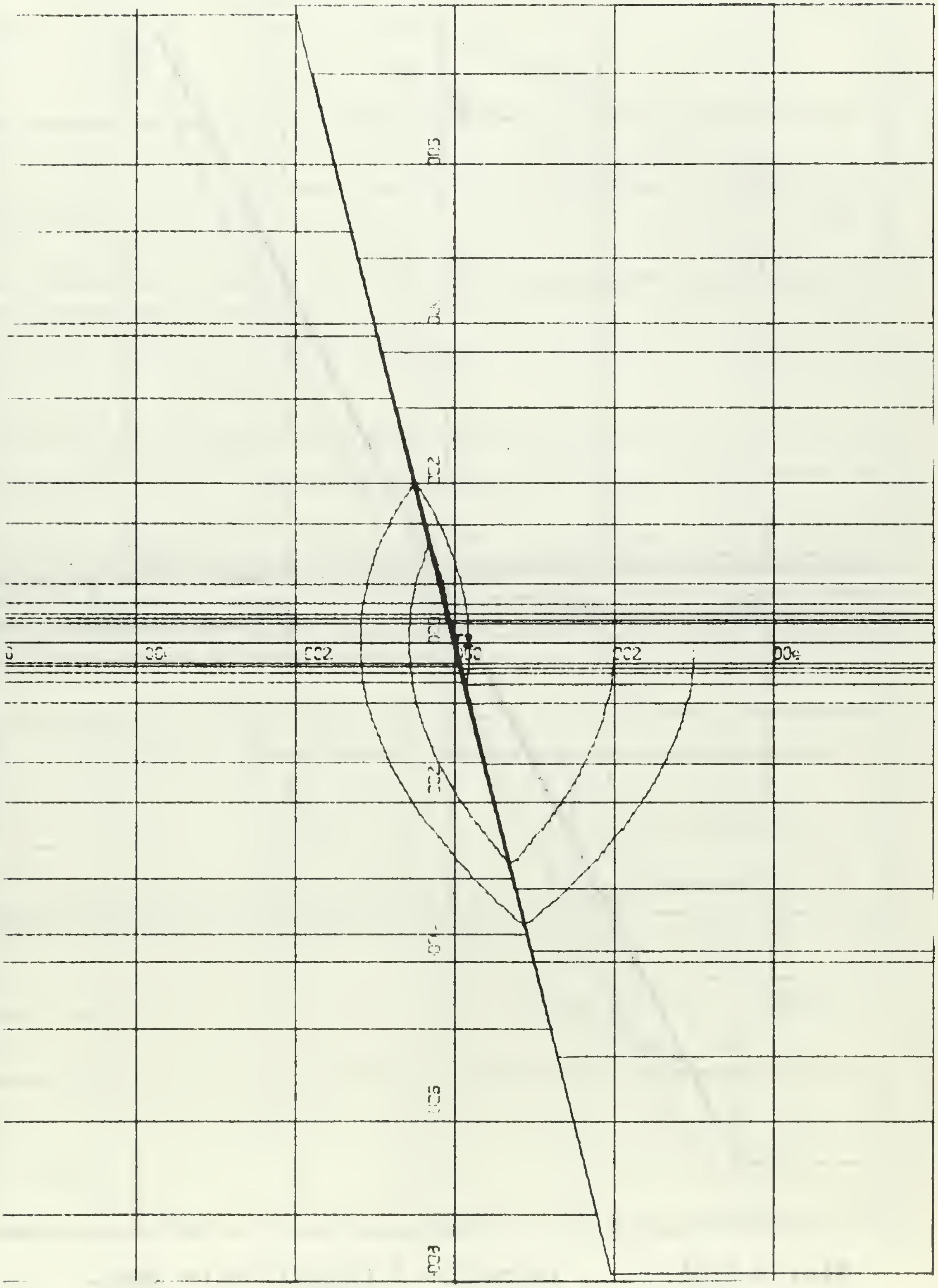


Fig. 5-8(e). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(f): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\ \mathrm{Sgn}(2x-y),\ \dot{x} = y$

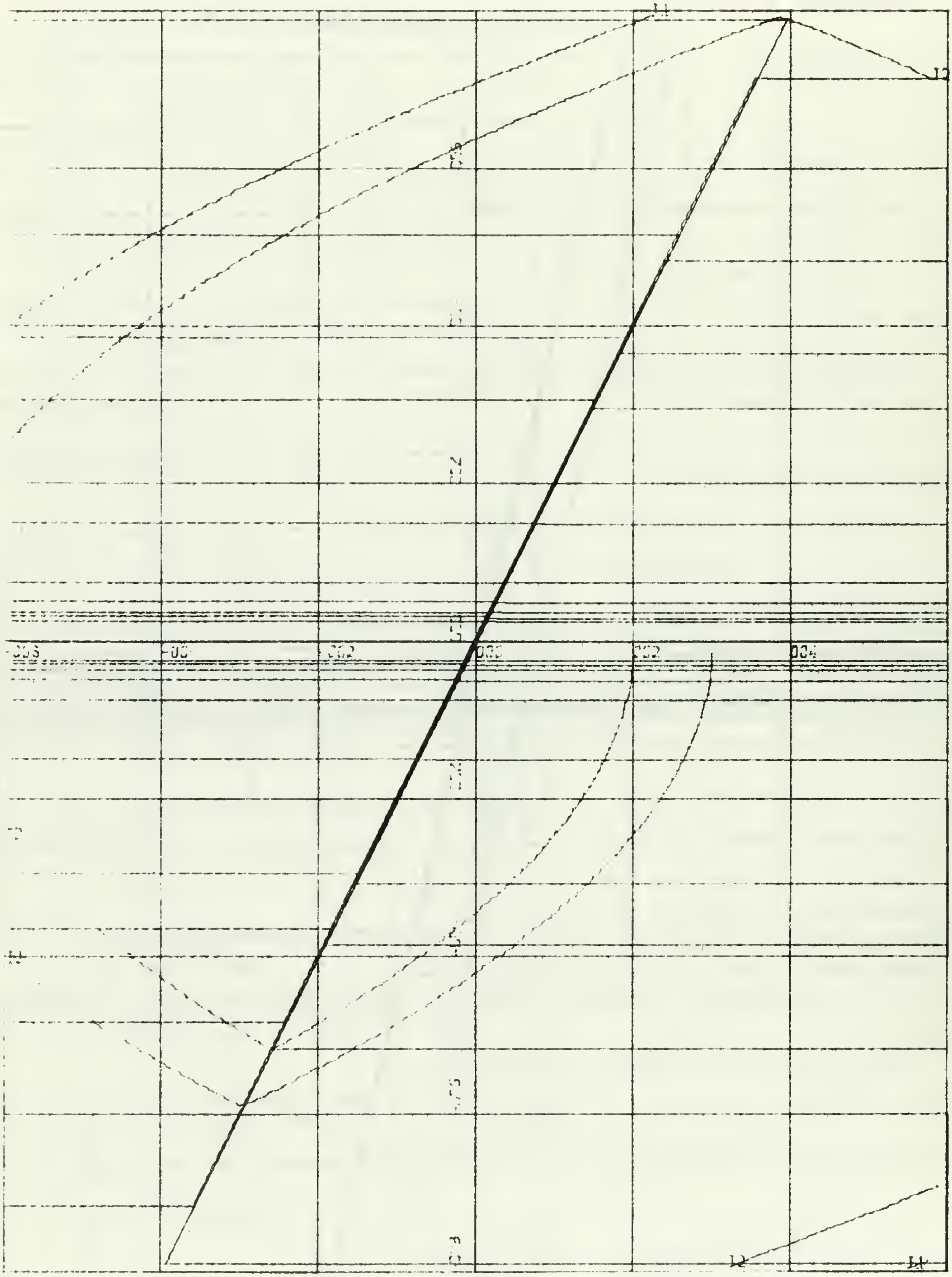


Fig. 5-8(f). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(g): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\ \mathrm{Sgn}(2x+y),\ \dot{x} = y$

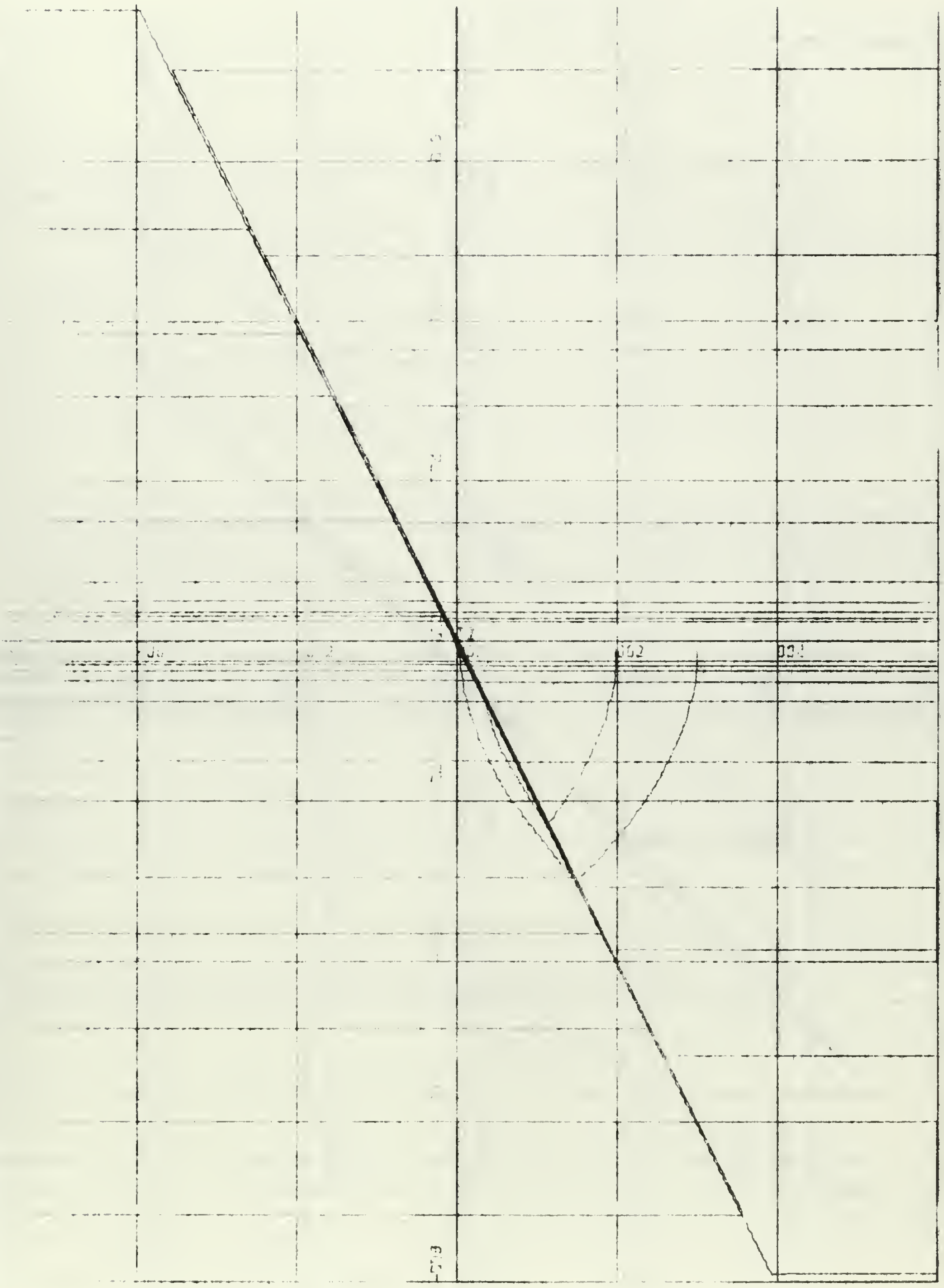


Fig. 5-8(g). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(h): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\,\mathrm{Sgn}(x-y),\ \dot{x} = y$

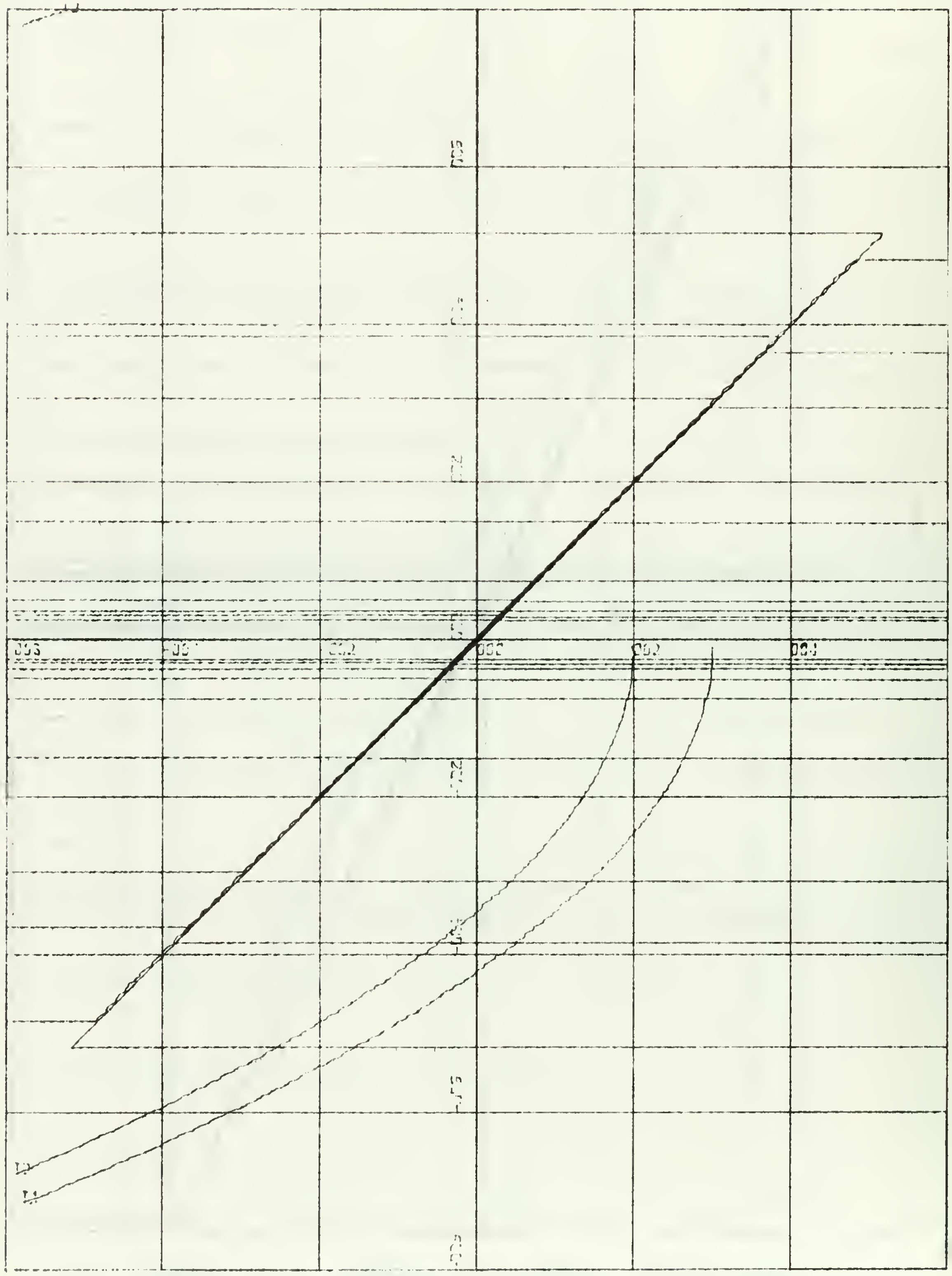


Fig. 5-8(h). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(i): $\ddot{x} + 0.02\dot{x} + u = 0$

$$u = -0.3\ \mathrm{Sgn}(x+y), \quad \dot{x} = y$$

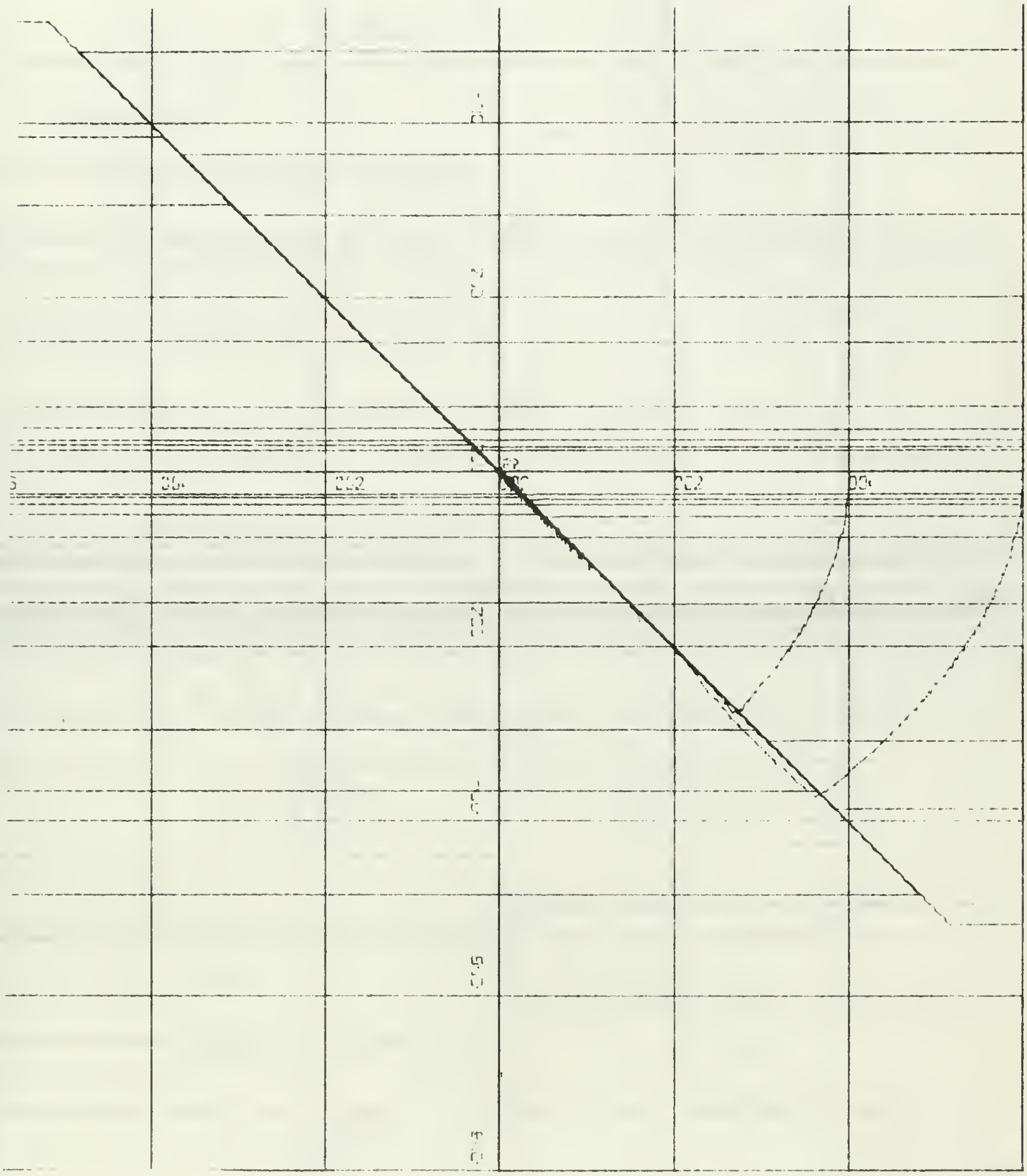


Fig. 5-8(i). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 5-8(j): $\ddot{x} + 0.02\dot{x} + u = 0$

$u = -0.3\ \mathrm{Sgn}(y), \quad \dot{x} = y$

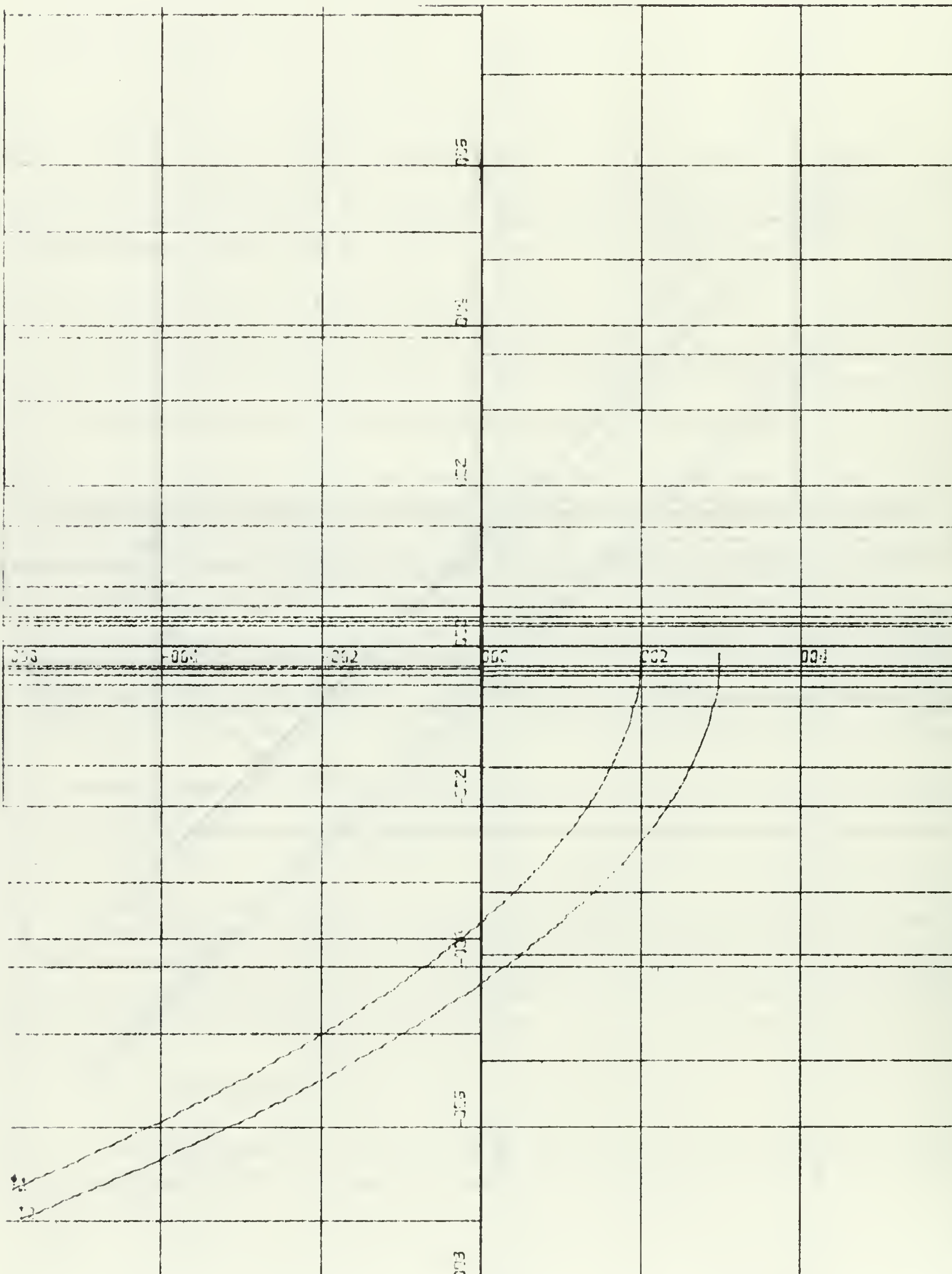


Fig. 5-8(j). X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

D. RELAY WITH DEAD ZONE (SEE PROGRAM 9)

For $\ddot{x} + \dot{x} + x + u = 0$

$$u = -K\,\mathrm{Sgn}(x)$$

Isoclines:

At dead zone region:

$$\ddot{x} + \dot{x} + x = 0$$

$$My + y + x = 0$$

$$y = \frac{-x}{M + 1}$$

At right:

$$\ddot{x} + \dot{x} + x - K = 0$$

$$My + y + x - K = 0$$

$$y = \frac{-x+K}{M + 1}$$

At left:

$$\ddot{x} + \dot{x} + x + K = 0$$

$$My + y + x + K = 0$$

$$y = \frac{-x-K}{M + 1}$$

Trajectories:

At dead zone region:

$$\ddot{x} + \dot{x} + x = 0, \quad \begin{cases} \mathrm{ZD\phi T(1)} = \mathrm{Z(2)} \\ \mathrm{ZD\phi T(2)} = -\mathrm{Z(2)} - \mathrm{Z(1)} \end{cases}$$

At right:

$$\ddot{x} + \dot{x} + x - K = 0, \quad \begin{cases} ZD\phi T(1) = Z(2) \\ ZD\phi T(2) = -Z(2) - Z(1) + K \end{cases}$$

At left:

$$\ddot{x} + \dot{x} + x + K = 0, \quad \begin{cases} ZD\phi T(1) = Z(2) \\ ZD\phi T(2) = -Z(2) - Z(1) - K \end{cases}$$

Example 9: $\ddot{x} + 0.2\dot{x} + x + u = 0$

$u = -0.3\ \mathrm{Sgn}(x)$

dividing lines: $x = \pm 0.2$

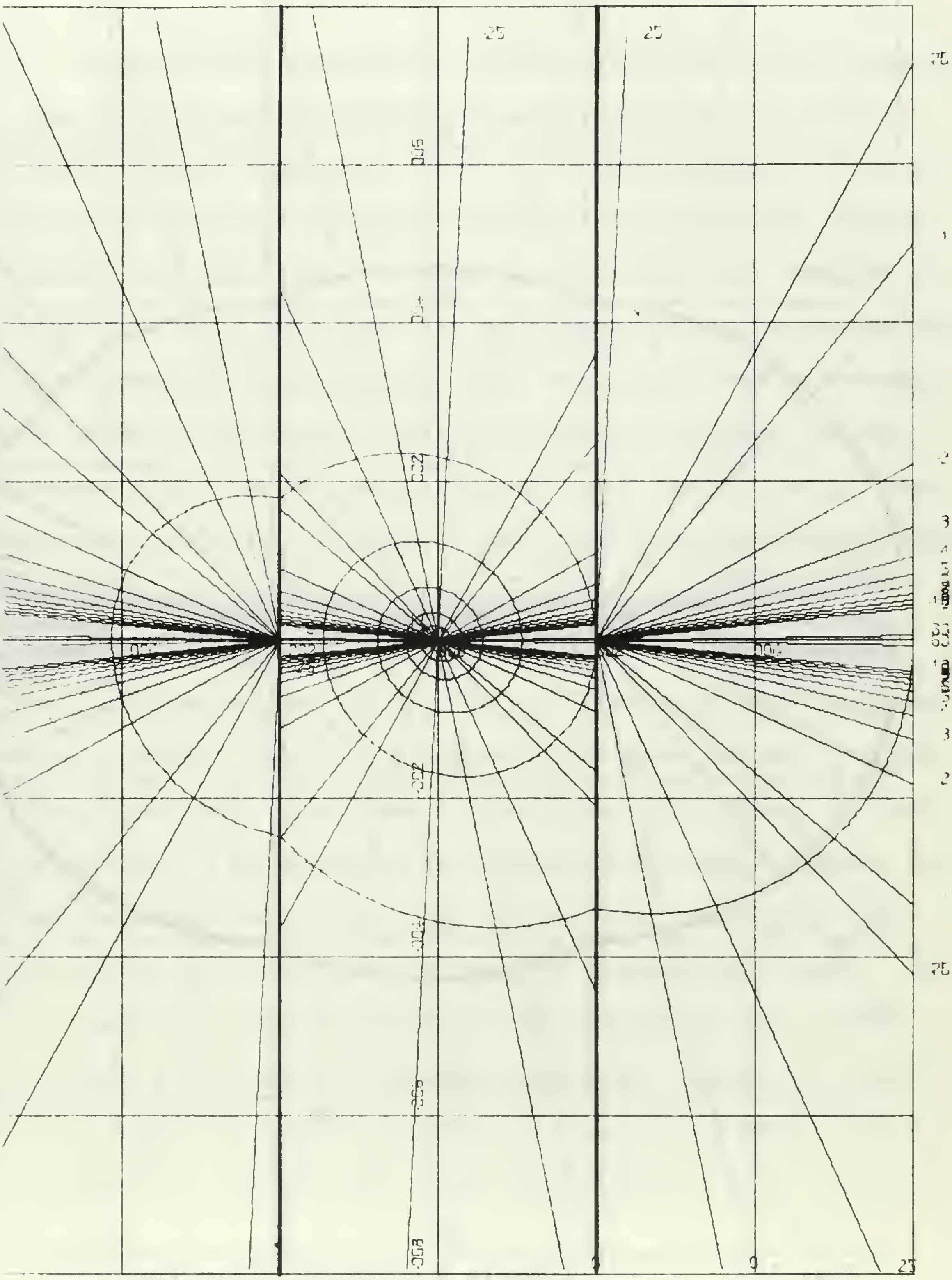


Fig. 9. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

Example 10: $\ddot{x} + x + u = 0$

$u = -0.3\ \mathrm{Sgn}(x)$

dividing lines: $x = \pm 0.2$

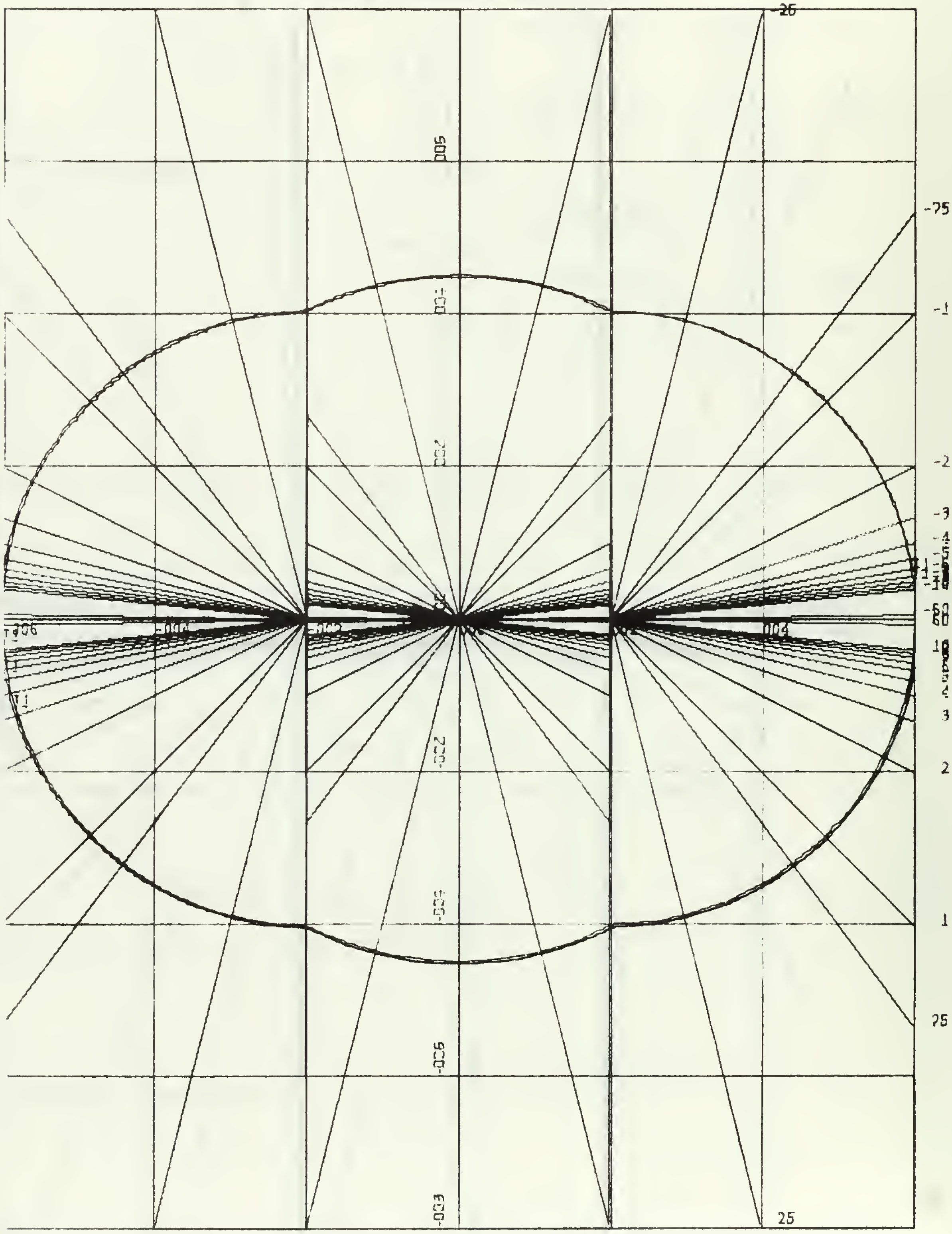


Fig. 10. X-Scale = 2.00E-01 Units Inch.
Y-Scale = 2.00E-01 Units Inch.

IV. CONCLUSIONS

Chapter III summarizes eleven useful methods. Some of them draw direction fields to get trajectories. Some of them are good supplements to the isocline method. Some of them, by using geometrical relationships, draw the trajectories step-by-step. In general, none of these methods is a perfect one. For specific problems one may be better than the others in terms of the labor required. But if we want to look at the overall picture of some nonlinear system characteristics such as stability limit cycles, etc., none of them can do this easily. All need lots of time and hard labor.

The isocline method has many advantages. Programs have been developed to compute and draw both isoclines and trajectories. All the programs in Chapter IV and V, first draw isoclines, then draw trajectories from given initial conditions. In analysis and design work, isoclines may not be needed for all problems. Conversely, sometimes only isoclines are needed and we do not need trajectories. Use of isoclines with superposed trajectories gives a global pitcure. That is, it indicates what can happen to trajectories for all initial conditions, and tells whether the system is stable or not, has limit cycle or not, ... etc.

The programs generated in Chapter V add dividing lines to the isoclines and trajectories. By changing or shifting dividing line position, the trajectories are shaped. This is helpful for analysis and design work.

In this thesis, all graphs are plotted by machine and all are two dimensional. The variable symbols y vs x or $\dot{x}$ vs x are used. There are no time variables in the plot. If the system equation involves time as a variable, we may choose the δ method to work it or apply some transformation, changing the equation into the standard type forms. In general, machine time required to compute the isoclines and trajectories of a second-order nonlinear differential equation was about 60 sec. This is an appreciable saving compared with hand labor.

APPENDIX

COMPUTER PROGRAMS

PROGRAM 1

```
C
C      PROGRAM 1
C
       INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1                'T8  ','T9  ','T10 '/
       INTEGER BL/'    '/
       REAL*8 ITITLE(12)
       REAL*4 MC,MD,ML,XP(900),Y(900),M
       EXTERNAL G,F
       REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
       CALL ERRSET(207,260,-1,1,0,209)
C
       READ(5,96) SCALE,NUR
    96 FORMAT(F10.0,I10)
       YMAX=SCALE*5.
       XO=-YMAX
       XMAX=SCALE*4.5
       XD=YMAX/400.
       XL=XMAX
       READ(5,98) ITITLE
    98 FORMAT(6A8)
       MC=1
       DO 10 I=1,NUR
       READ(5,99) MO,MD,ML
    99 FORMAT(4F10.0)
       M=MO
     2 X=XO
       L=1
     3 DN=M+F(X)
       IF(DN) 30,41,30
    30 Y(L)=-G(X)/DN
       CALL OVERFL(JA)
       IF(JA .NE. 2) GO TO 41
       IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT. -YMAX)) GO TO 41
       IF(L .GE. 900) GO TO 4
       XP(L)=X
       IF(X .GE. XL) GO TO 4
       L=L+1
       GO TO 43
    41 IF(X .GE. XL) GO TO 45
       L=L-1
       ASSIGN 442 TO IGO
       GO TO 440
    43 X=X+XD
       GO TO 3
```

```
    4 ASSIGN 42 TO IGO
  440 IF(L .LE. 1) GO TO 441
C     WRITE(6,97) (XP(J),Y(J),J=1,L)
      MB=ABS(M)+1.E-7
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
      MC=2
   97 FORMAT(1H0/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 CALL DE(Z,ZDOT)
      S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -YMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
```

```
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1          Z000000F3,Z000000F4,Z000000F5,
     2          Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
```

```
    3 LAB = LAB + 25o**(I)*FL
    5 RETURN
      END
      REAL FUNCTION G(X)
      G=X
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=0.5
      RETURN
      END


      SUBROUTINE  DE( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-0.5*Z(2)-Z(1)
      RETURN
      END
INPUT DATA
       10.          3
'BOX H Y.L.HSIUNG                   JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION     '
      -20.         2.         20.
       -1.        0.25         1.
     -100.        40.        100.
     0.0625        4
        0.         8.         20.        10.
        0.        15.        -30.         0.
        0.        14.        -40.         0.
        0.         8.         20.        10.
```

PROGRAM 2

```
C
C     PROGRAM 2
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1              'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MC,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDCT(2),T,TO,TD,TL
      CALL ERRSET(207,256,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XD=YMAX/400.
      XMAX=SCALE*JX
      XO=-XMAX
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MC,MD,ML
   99 FORMAT(4F10.0)
      DO 100  JD=1,2
      M=MC
    2 X=XO
      L=1
    3 DISC=M**2-4.*G(X)
      IF(DISC) 41,30,30
   30 DISC=SQRT(DISC)
      IF(JD .EQ. 1)  DISC=-DISC
      Y(L)=(-M+DISC)/2.
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
      IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GO TO 4
      L=L+1
```

```
      GO TO 43
   41 IF(X .GE. XL) GO TO 45
      L=L-1
      ASSIGN 442 TO IGO
      GO TO 440
   43 X=X+XD
      GO TO 3
    4 ASSIGN 42 TO IGO
  440 IF(L .LE. 1) GO TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L)
      MB=ABS(M)+1.E-7
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FORMAT(1HO/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 100
      M=M+MD
      GO TO 2
  100 CONTINUE
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TO,NS
      DO 29  I=1,NS
      READ(5,99) TC,TL,Z(1),Z(2)
      IF(TL .LE. TC) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 CALL DE(Z,ZDOT)
      S=RKLDEQ(2,Z,ZDOT,T,TC,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
      WRITE(6,88) T,Z,ZDOT,L,JA,JB
   88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -YMAX)) GO TO 231
```

```
      IF((Z(2) .GT. YMAX) .CR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL CRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GC TC 24
  240 IGO=1
      GC TC 24
   29 CONTINUE
  291 LB=BL
      CALL CRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBRCUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1      Z000000F3,Z000000F4,Z000000F5,
     2      Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GC TC 4
      LAB = STARS
      GC TC 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      CC 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FCRMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
```

```
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTION F(X)
      F=(1.-X**2)
      RETURN
      END
      REAL FUNCTION G(X)
      G=X*ABS(X)
      RETURN
      END
C
      SUBROUTINE  DE( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-Z(2)**2-Z(1)*DABS(Z(1))
      RETURN
      END
INPUT DATA
   0.5          3
'BOX H Y.L.HSIUNG                     JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION    '
   -100.      200.      100.
    -10.        1.       10.
   -0.75        .5       .75
   0.0625       4
      0.        8.        1.        1.
      0.       10.      -1.5        2.
      0.        8.      -.75       .75
      0.        8.        .5        .5
```

PROGRAM 3

```
C
C     PROGRAM 3
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1              'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MO,MD,ML,XP(900),Y(900),M,XT(900)
      REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*5.
      YO=-YMAX
      XMAX=SCALE*4.5
      YD=YMAX/400.
      YL=YMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 YA=YO
      L=1
    3 CONTINUE
      XT(L)=-M*YA/(1.+ABS(YA))
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
      IF((XT(L) .GT. XMAX) .OR. (XT(L) .LT. -XMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=XT(L)
      Y(L)=YA
      IF(YA .GE. YL) GO TO 4
      L=L+1
      GO TO 43
   41 IF(YA .GE. YL) GO TO 45
      L=L-1
      ASSIGN 442 TO IGO
      GO TO 440
   43 YA=YA+YD
      GO TO 3
    4 ASSIGN 42 TO IGO
```

```
440 IF(L .LE. 1) GO TO 441
    WRITE(6,97) (XP(J),Y(J),J=1,L)
    MB=ABS(M)+1.E-7
    IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
    IF(M .LT. 0) MB=-MB
    CALL LABEL(MB,LB)
    CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
    MC=2
 97 FORMAT(1H0/(8E15.6))
441 GO TO IGO, (42,442)
442 L=1
    YA=YA+YD
    GO TO 3
 42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
    M=M+MD
    GO TO 2
 10 CONTINUE
    GO TO 20
 45 L=L-1
    GO TO 4
 20 READ(5,96) TD,NS
    DO 29  I=1,NS
    READ(5,99) TO,TL,Z(1),Z(2)
    IF(TL .LE. TO) GO TO 291
    T=TO
232 NT=0
    L=0
 22 CALL DE(Z,ZDOT)
    S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
    IF(S-1.) 21,22,23
 21 STOP
 23 CALL OVERFL(JA)
    IF(JA .NE. 2) GO TO 240
    CALL DVCHK(JB)
    IF(JB .NE. 2) GO TO 240
    WRITE(6,88) T,Z,ZDOT,L,JA,JB
 88 FORMAT(10X,5D14.5,3I5)
    IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -YMAX)) GO TO 231
    IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
    IGO=1
    L=L+1
    XP(L)=Z(1)
    Y(L)=Z(2)
    IF(L .GE. 900) GO TO 24
230 IF(T .LT. TL) GO TO 22
 24 IF(L .LE. 1) GO TO 290
    WRITE(6,97) (XP(J),Y(J),J=1,L),T
    LB=LT(I)
```

```
      CALL CRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL CRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,5,5,2,2,9,10,1,LAST)
      STOP
      END



      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1       Z000000F3,Z000000F4,Z000000F5,
     2       Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
```

```
    5 RETURN
      END
      REAL FUNCTION G(X)
      RETURN
      END
C
      REAL FUNCTION F(X)
      RETURN
      END



      SUBROUTINE  DE( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-Z(1)*DABS(Z(2))-Z(1)
      RETURN
      END
INPUT DATA
       0.8         3
'BOX H Y.L.HSIUNG                           JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION         '
       -1.        .2         1.
      -10.        4.        10.
      -12.       24.        12.
     .0625        4
        0.        8.        -2.         0.
        0.       12.      -2.5         -2.
        0.       14.       2.5          0.
        0.       10.        2.       -1.6
```

```
PROGRAM 4
C
C     PROGRAM 4
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1               'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MO,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      READ(5,99) C1,C2
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XMAX=SCALE*JX
      XO=-XMAX
      XD=YMAX/400.
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 X=XO
      L=1
    3 IF((X .GT. -C1) .AND. (X .LE. C1) ) GO TO 31
      DN=M+C2
      IF(DN) 32,41,32
   32 Y(L)=-C1/DN
      IF(X .LE.  -C1) Y(L)=-Y(L)
      GO TO 33
   31 DN=M+F(X)
      IF(DN) 30,41,30
   30 Y(L)=-G(X)/DN
   33 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
      IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
```

```
      IF(X .GE. XL) GO TO 4
      L=L+1
      GO TO 43
   41 IF(X .GE. XL) GO TO 45
      L=L-1
      ASSIGN 442 TO IGO
      GO TO 440
   43 X=X+XD
      GO TO 3
    4 ASSIGN 42 TO IGO
  440 IF(L .LE. 1) GO TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-3
      IF(MB .LT. 10.) MB=10.*ABS(M)+1.E-3
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-3
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FORMAT(1HO/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 IF(Z(1) .GT. -C1) GO TO 220
      CALL DL(Z,ZDOT)
      GO TO 221
  220 IF(Z(1) .GT. C1) GO TO 222
      CALL DC(Z,ZDOT)
      GO TO 221
  222 CALL DR(Z,ZDOT)
  221 S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
```

```
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1        Z000000F3,Z000000F4,Z000000F5,
     2        Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
```

```
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
      WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTION G(X)
      G=X
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=0.2
      RETURN
      END



      SUBROUTINE DC(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-0.2*Z(2)-Z(1)
      RETURN
      END
```

```
      SUBROUTINE DR(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-0.2*Z(2)+.2
      RETURN
      END



      SUBROUTINE DL(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-0.2*Z(2)-.2
      RETURN
      END
INPUT DATA
        .2         3
        .2        .2
'BOX H Y.L.HSIUNG                     JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION    '
       -.5        .1        .5
      -6.3        .6       6.3
      -7.5        5.       7.5
     .0625         3
        0.        30.        1.        0.
        0.        20.        .8        0.
        0.        20.        .4        0.
```

```
PROGRAM 5
C
C      PROGRAM 5
C
       INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1                'T8  ','T9  ','T10 '/
       INTEGER BL/'    '/
       REAL*8 ITITLE(12)
       REAL*4 MO,MD,ML,XP(900),Y(900),M,XQ(2),YQ(2)
       EXTERNAL G,F
       REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
       CALL ERRSET(207,260,-1,1,0,209)
C
       READ(5,96) SCALE,NUR
       READ(5,99) A,B,C
       IX=6
       IY=8
       JX=(IX+1)/2
       JY=(IY+1)/2
    96 FORMAT(F10.0,I10)
       YMAX=SCALE*JY
       XMAX=SCALE*JX
       XO=-XMAX
       XD=YMAX/400.
       XL=XMAX
       READ(5,98) ITITLE
    98 FORMAT(6A8)
       MC=1
       DO 10 I=1,NUR
       READ(5,99) MO,MD,ML
    99 FORMAT(4F10.0)
       M=MO
     2 XX=(M+1.)*C/(M+1.-B)
       X=-ABS(XX)
       CALL DVCHK(JB)
       IF(JB .NE. 2) X=-XMAX
       CALL OVERFL(JA)
       IF(JA .NE. 2) X=-XMAX
       X=AMAX1(X,-XMAX)
       XO=X
       XL=-X
       L=1
     3 CONTINUE
    31 DN=M+F(X)
       IF(DN) 30,41,30
    30 Y(L)=-G(X)/DN
    33 CALL OVERFL(JA)
       IF(JA .NE. 2) GO TO 41
```

```
      IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT. -YMAX)) GO TO 41
      IF((X .GT. XMAX) .OR. (X .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GO TO 4
      L=L+1
      GO TO 43
   41 IF(X .GE. XL) GO TO 45
      L=L-1
      ASSIGN 442 TO IGO
      GO TO 440
   43 X=X+XD
      GO TO 3
    4 ASSIGN 42 TO IGO
  440 IF(L .LE. 1) GO TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-3
      IF(MB .LT. 10.) MB=10.*ABS(M)+1.E-3
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-3
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      IF(XP(1) .EQ. -XMAX) GO TO 445
      IF(XX .GT. 0) GO TO 465
      XQ(1)=-XMAX
      XQ(2)=XP(L)
      YQ(1)=Y(L)
      YQ(2)=Y(L)
      GO TO 466
  465 XQ(1)=-XMAX
      XQ(2)=XO
      YQ(1)=Y(1)
      YQ(2)=Y(1)
  466 CALL DRAW(2,XQ,YQ,MC,0,BL,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1LAST)
      MC=2
  445 LQ=BL
      IF(XP(L) .EQ. XMAX) LQ=LB
      CALL DRAW(L,XP,Y,MC,0,LQ,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      IF(XP(L) .EQ. XMAX) GO TO 441
      IF(XX .GT. 0) GO TO 467
      XQ(1)=XO
      XQ(2)=XMAX
      YQ(1)=Y(1)
      YQ(2)=Y(1)
      GO TO 468
  467 XQ(1)=XP(L)
      XQ(2)=XMAX
```

```
      YQ(1)=Y(L)
      YQ(2)=Y(L)
  468 CALL DRAW(2,XQ,YQ,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1LAST)
      MC=2
   97 FORMAT(1HO/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 IF(Z(1) .LT. (C-B*Z(2))/A) GO TO 220
      CALL DR(Z,ZDOT)
      GO TO 221
  220 IF(Z(1) .LT. (-C-B*Z(2))/A)  GO TO 222
      CALL DC(Z,ZDOT)
      GO TO 221
  222 CALL DL(Z,ZDOT)
    221 S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
```

```
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1      Z000000F3,Z000000F4,Z000000F5,
     2      Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
```

```
      IB=IA+1
      CO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTION G(X)
      G=X
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=1.
      RETURN
      END



      SUBROUTINE DC(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-Z(2)-Z(1)
      RETURN
      END



      SUBROUTINE DR(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-Z(2)+.3
      RETURN
      END



      SUBROUTINE DL(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-Z(2)-.3
      RETURN
      END
INPUT DATA
         .2          3
         0.         .8          .3
'BOX H Y.L.HSIUNG                            JULY 1969'
```

```
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION    '
     -.5        .1        .5
   -6.3        .6       6.3
   -7.5        5.       7.5
  .0625         3
     0.       30.        1.        0.
     0.       20.        .8        0.
     0.       20.        .4        0.
```

```
C
C     PROGRAM 6
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1               'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MC,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      COMMON C1
      REAL*8 Z(2),ZDOT(2),T,TC,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      READ(5,99) C1,C2
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XMAX=SCALE*JX
      XO=-XMAX
      XD=YMAX/400.
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MC,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 X=XO
      L=1
    3 IF((X .GT. -C1) .AND. (X .LE. C1)) GO TO 35
      GO TO 31
   35 DN=M+C2
      IF(DN) 32,41,32
   32 Y(L)=-C1/DN
      IF(X .LE.  -C1) Y(L)=-Y(L)
      GO TO 33
   31 DN=M+F(X)
      IF(DN) 30,41,30
   30 Y(L)=-G(X)/DN
   33 CALL OVERFL(JA)
```

```
      IF(JA .NE. 2) GC TO 41
      IF((Y(L) .GT. YMAX) .CR. (Y(L) .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GC TO 4
      L=L+1
      GC TC 43
   41 IF(X .GE. XL) GO TO 45
      L=L-1
      ASSIGN 442 TC IGO
      GO TC 440
   43 X=X+XD
      GO TC 3
    4 ASSIGN 42 TC IGC
  440 IF(L .LE. 1) GC TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-3
      IF(MB .LT. 10.) MB=10.*ABS(M)+1.E-3
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-3
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FORMAT(1H0/(8E15.6))
  441 GC TC IGC, (42,442)
  442 L=1
      X=X+XD
      GC TC 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GC TC 2
   10 CONTINUE
      GO TC 20
   45 L=L-1
      GC TC 4
   2C READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TC,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TC 291
      T=TC
  232 NT=0
      L=0
   22 IF((Z(1) .GT. -C1) .AND. (Z(1) .LT. C1)) GO TO 22C
      CALL CE(Z,ZDOT)
      GC TC 221
  22C CALL CI(Z, ZDCT)
  221 S=RKLDEQ(2,Z,ZDCT,T,TC,NT)
```

```
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL CVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL CVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1        Z000000F3,Z000000F4,Z000000F5,
     2        Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
```

```
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GC TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTICN G(X)
      COMMON C1
      IF(X .LE. 0.) GO TO 1
      G=X-C1
      RETURN
    1 G=X+C1
      RETURN
      END
C
      REAL FUNCTICN F(X)
      F=0.2
      RETURN
      END



      SUBRCUTINE  DE( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)-Z(1)
      RETURN
      END
```

```
      SUBROUTINE DI(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)
      RETURN
      END
INPUT DATA
         .2          3
         .3         .2
'BOX H Y.L.HSIUNG                             JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION           '
        -.5         .1         .5
       -6.3         .6        6.3
       -7.5         5.        7.5
      .0625          3
         0.        30.         1.         0.
         0.        20.         .8         0.
         0.        20.         .4         0.
```

PROGRAM 7

```
C
C     PROGRAM 7
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1               'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MO,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDOT(2),T,TC,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XD=YMAX/400.
      XMAX=SCALE*JX
      XO=-XMAX
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 X=XO
      L=1
    3 DN=M+F(X)
      IF(DN) 30,41,30
   30 Y(L)=-G(X)/DN
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
      IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GO TO 4
      L=L+1
      GO TO 43
   41 IF(X .GE. XL) GO TO 45
      L=L-1
```

```
      ASSIGN 442 TO IGO
      GO TO 440
   43 X=X+XD
      GO TO 3
    4 ASSIGN 42 TO IGO
  440 IF(L .LE. 1) GO TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-7
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1           LAST)
      MC=2
   97 FORMAT(1H0/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TO,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 231
      T=TO
  232 NT=0
      L=0
   22 IF(Z(1) .LE. 0.) GO TO 222
      CALL DR(Z,ZDOT)
      GO TO 223
  222 CALL DL(Z,ZDOT)
  223 S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
```

```
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL CRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1      Z000000F3,Z000000F4,Z000000F5,
     2      Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
```

```
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTION G(X)
      IF(X .LE. 0)  GO TO 1
      G=.3
      RETURN
    1 G=-.3
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=.2
      RETURN
      END



      SUBROUTINE CR(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)-.3
      RETURN
      END



      SUBROUTINE CL(Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)+.3
      RETURN
      END
INPUT DATA
        .2        3
'BOX H Y.L.HSIUNG                           JULY 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION           '
      -10.          1.          10.
```

-0.75	0.5	.75	
-60.	120.	60.	
.0325	1		
0.	20.	.6	0.

```
C
C
C     PROGRAM 8
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1              'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MO,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      READ(5,99) A,B
      TANA=A/B
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XD=YMAX/400.
      XMAX=SCALE*JX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      L=4
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 XP(1)=-XMAX
      Y(1)=-.3/(M+F(X))
      XP(2)=-B/A*Y(1)
      Y(2)=Y(1)
      XP(3)=-XP(2)
      Y(3)=-Y(1)
      XP(4)=XMAX
      Y(4)=Y(3)
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 42
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 42
      DO 287 IA=1,L
```

```
      IF((XP(IA) .GT. XMAX) .OR. (XP(IA) .LT. -XMAX)) GO TO 42
      IF(( Y(IA) .GT. YMAX) .OR. ( Y(IA) .LT. -YMAX)) GO TO 42
  287 CONTINUE
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-3
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-3
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FORMAT(1H0/(8E15.6))
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 YC=-Z(1)*TANA
      IF(Z(2) .LE. YC) GO TO 222
      CALL DR(Z,ZDOT)
      GO TO 223
  222 CALL DL(Z,ZDOT)
  223 S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
   88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
```

```
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1         Z000000F3,Z000000F4,Z000000F5,
     2         Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
```

```
    5 RETURN
      END
      REAL FUNCTION G(X)
      G=0.
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=.02
      RETURN
      END



      SUBROUTINE  DR( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.02*Z(2)+.3
      RETURN
      END



      SUBROUTINE  DL( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.02*Z(2)-.3
      RETURN
      END
INPUT DATA
        .2          3
        4.        -1.
'BOX H Y.L.HSIUNG                      JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION     '
      -1.          .2          1.
     -10.          2.         10.
     -12.         24.         12.
     .0325          2
        0.        20.          .2          0.
        0.        20.          .3          0.
```

```
C
C
C     PROGRAM 8-A
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1              'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MC,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      READ(5,99) A,B
      TANA=B/A
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XD=YMAX/400.
      XMAX=SCALE*JX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      L=4
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 XP(1)=-XMAX
      Y(1)=-.3/(M+F(X))
      XP(2)=-B/A*Y(1)
      Y(2)=Y(1)
      XP(3)=-XP(2)
      Y(3)=-Y(1)
      XP(4)=XMAX
      Y(4)=Y(3)
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 42
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 42
      DO 287 IA=1,L
```

```
      IF((XP(IA) .GT. XMAX) .OR. (XP(IA) .LT. -XMAX)) GO TO 42
      IF(( Y(IA) .GT. YMAX) .OR. ( Y(IA) .LT. -YMAX)) GO TO 42
  287 CONTINUE
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-3
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-3
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FORMAT(1H0/(8E15.6))
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 XC=-Z(2)*TANA
      IF(Z(1) .LE. XC) GO TO 222
      CALL DR(Z,ZDOT)
      GO TO 223
  222 CALL DL(Z,ZDOT)
  223 S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL CVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL CVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
   88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
```

```
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1          Z000000F3,Z000000F4,Z000000F5,
     2          Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
```

```
    5 RETURN
      END
      REAL FUNCTION G(X)
      G=0.
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=.02
      RETURN
      END



      SUBROUTINE  DR( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.02*Z(2)-.3
      RETURN
      END



      SUBROUTINE  DL( Z,ZDOT)
      REAL*8 Z(2),ZDOT(2)
      ZDOT(1)=Z(2)
      ZDOT(2)=-.02*Z(2)+.3
      RETURN
      END
INPUT DATA
        .2          3
        1.          1.
'BOX H Y.L.HSIUNG                                JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION               '
       -1.          .2          1.
      -10.          2.          10.
      -12.          24.         12.
     .0325          2
        0.          20.          .3          0.
        0.          20.          .2          0.
```

```
C
C     PROGRAM 9
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1               'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MC,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      COMMON C1,AK
      REAL*8 Z(2),ZDOT(2),T,TC,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
      READ(5,99) C1,AK
      IX=6
      IY=8
      JX=(IX+1)/2
      JY=(IY+1)/2
   96 FORMAT(F10.0,I10)
      YMAX=SCALE*JY
      XD=YMAX/400.
      XMAX=SCALE*JX
      XO=-XMAX
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MC,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 X=XO
      L=1
    3 IF((X .GT. -C1) .AND. (X .LE. C1)) GO TO 35
      GO TO 31
   35 DN=M+F(X)
      IF(DN) 32,41,32
   32 Y(L)=-X/DN
      GO TO 33
   31 DN=M+F(X)
      IF(DN) 30,41,30
   30 Y(L)=-G(X)/DN
   33 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
```

```
      IF((Y(L) .GT. YMAX) .CR. (Y(L) .LT. -YMAX)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GC TO 4
      L=L+1
      GC TC 43
   41 IF(X .GE. XL) GC TO 45
      L=L-1
      ASSIGN 442 TO IGO
      GC TC 440
   43 X=X+XD
      GC TC 3
    4 ASSIGN 42 TC IGC
  440 IF(L .LE. 1) GC TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-7
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,
     1          LAST)
      MC=2
   97 FCRMAT(1HO/(8E15.6))
  441 GO TC IGO, (42,442)
  442 L=1
      X=X+XD
      GC TC 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GC TO 10
      M=M+MD
      GO TC 2
   10 CONTINUE
      GO TC 20
   45 L=L-1
      GO TC 4
   20 READ(5,96) TC,NS
      DC 29  I=1,NS
      READ(5,99) TC,TL,Z(1),Z(2)
      IF(TL .LE. TC) GC TC 291
      T=TO
  232 NT=0
      L=0
   22 IF(Z(1) .GT. -C1) GO TC 220
      CALL DL(Z,ZCCT)
      GC TC 221
  220 IF(Z(1) .GT. C1) GO TC 222
      CALL DC(Z,ZCCT)
      GC TC 221
  222 CALL DR(Z,ZCCT)
```

```
  221 S=RKLDEQ(2,Z,ZDOT,T,TC,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL CVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT. -XMAX)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT. -YMAX)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,JY,JX,2,2,IX,IY,1,LAST)
      STOP
      END



      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1        Z000000F3,Z000000F4,Z000000F5,
     2        Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
```

```
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
C     WRITE(6,9) LAB
C
      IF (IA.GE.3) GO TO 5
      IB=IA+1
      DO 3 I=IB, 3
    3 LAB = LAB + 256**(I)*BL
    5 RETURN
      END
      REAL FUNCTION G(X)
      COMMON C1,AK
      IF(X .LE. 0.) GO TO 1
      G=X-AK
      RETURN
    1 G=X+AK
      RETURN
      END
C
      REAL FUNCTION F(X)
      F=.2
      RETURN
      END
```

```
      SUBROUTINE CR(Z,ZDCT)
      REAL*8 Z(2),ZDOT(2)
      COMMON C1,AK
      ZDOT(1)=Z(2)
      ZDCT(2)=-.2*Z(2)-Z(1)+AK
      RETURN
      END


      SUBROUTINE CL(Z,ZDCT)
      REAL*8 Z(2),ZDOT(2)
      COMMON C1,AK
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)-Z(1)-AK
      RETURN
      END


      SUBROUTINE CC(Z,ZDCT)
      REAL*8 Z(2),ZDCT(2)
      COMMON C1,AK
      ZDOT(1)=Z(2)
      ZDOT(2)=-.2*Z(2)-Z(1)
      RETURN
      END
INPUT CATA
        .2          3
        .2         .2
'ROX H Y.L.HSIUNG                        JULY 1969'
'PHASE PORTRAIT CF 2ND ORCER DIFF. EQUATION      '
      -10.          1.         10.
     -C.75         0.5         .75
      -60.        120.         60.
     .C625           1
        0.         40.          .6          0.
```

PROGRAM 10

```
C  IF YOU WANT TO SHFIT THE CENTER OF COORDINATES,PLEASE,
C TO COMPARE THIS PROGRAM.
C
C      PROGRAM 10
C
C
      INTEGER LT(10)/'T1  ','T2  ','T3  ','T4  ','T5  ','T6  ','T7  ',
     1               'T8  ','T9  ','T10 '/
      INTEGER BL/'    '/
      REAL*8 ITITLE(12)
      REAL*4 MO,MD,ML,XP(900),Y(900),M
      EXTERNAL G,F
      REAL*8 Z(2),ZDOT(2),T,TO,TD,TL
      CALL ERRSET(207,260,-1,1,0,209)
C
      READ(5,96) SCALE,NUR
   96 FORMAT(F10.0,I10)
      XMIN=-1.*SCALE
      YMIN=-1.*SCALE
      YMAX=SCALE*9.0
      XO=XMIN
      XMAX=SCALE*8.5
      XD=YMAX/400.
      XL=XMAX
      READ(5,98) ITITLE
   98 FORMAT(6A8)
      MC=1
      DO 10 I=1,NUR
      READ(5,99) MO,MD,ML
   99 FORMAT(4F10.0)
      M=MO
    2 X=XO
      L=1
    3 DN=M+F(X)
      IF(DN) 30,41,30
   30 Y(L)=-G(X)/DN
      CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 41
      IF((Y(L) .GT. YMAX) .OR. (Y(L) .LT.  YMIN)) GO TO 41
      IF(L .GE. 900) GO TO 4
      XP(L)=X
      IF(X .GE. XL) GO TO 4
      L=L+1
      GO TO 43
   41 IF(X .GE. XL) GO TO 45
```

```
      L=L-1
      ASSIGN 442 TO IGO
      GO TO 440
   43 X=X+XD
      GO TO 3
    4 ASSIGN 42 TC IGO
  440 IF(L .LE. 1) GO TO 441
      WRITE(6,97) (XP(J),Y(J),J=1,L),M
      MB=ABS(M)+1.E-7
      IF(MB .LT. 1.) MB=100.*ABS(M)+1.E-7
      IF(M .LT. 0) MB=-MB
      CALL LABEL(MB,LB)
      CALL DRAW(L,XP,Y,MC,0,LB,ITITLE,SCALE,SCALE,1,1,2,2,9,10,1,LAST)
      MC=2
   97 FORMAT(1HO/(8E15.6))
  441 GO TO IGO, (42,442)
  442 L=1
      X=X+XD
      GO TO 3
   42 IF(ABS(M-ML) .LE. 1.E-3) GO TO 10
      M=M+MD
      GO TO 2
   10 CONTINUE
      GO TO 20
   45 L=L-1
      GO TO 4
   20 READ(5,96) TD,NS
      DO 29  I=1,NS
      READ(5,99) TO,TL,Z(1),Z(2)
      IF(TL .LE. TO) GO TO 291
      T=TO
  232 NT=0
      L=0
   22 CALL DE(Z,ZDOT)
      S=RKLDEQ(2,Z,ZDOT,T,TD,NT)
      IF(S-1.) 21,22,23
   21 STOP
   23 CALL OVERFL(JA)
      IF(JA .NE. 2) GO TO 240
      CALL DVCHK(JB)
      IF(JB .NE. 2) GO TO 240
C     WRITE(6,88) T,Z,ZDOT,L,JA,JB
C  88 FORMAT(10X,5D14.5,3I5)
      IF((Z(1) .GT. XMAX) .OR. (Z(1) .LT.  XMIN)) GO TO 231
      IF((Z(2) .GT. YMAX) .OR. (Z(2) .LT.  YMIN)) GO TO 231
      IGO=1
      L=L+1
      XP(L)=Z(1)
```

```
      Y(L)=Z(2)
      IF(L .GE. 900) GO TO 24
  230 IF(T .LT. TL) GO TO 22
   24 IF(L .LE. 1) GO TO 290
      WRITE(6,97) (XP(J),Y(J),J=1,L),T
      LB=LT(I)
      CALL DRAW(L,XP,Y,2,0,LB,ITITLE,SCALE,SCALE,1,1,2,2,9,10,1,LAST)
  290 GO TO (29,232),IGO
  231 IGO=2
      IF(T .GE. TL) IGO=1
      GO TO 24
  240 IGO=1
      GO TO 24
   29 CONTINUE
  291 LB=BL
      CALL DRAW(2,XP,Y,3,0,LB,ITITLE,SCALE,SCALE,1,1,2,2,9,10,1,LAST)
      STOP
      END


      SUBROUTINE LABEL (K,LAB)
      INTEGER M/Z00000060/,BL/Z00000040/,STARS/'****'/
      INTEGER N(10)/Z000000F0,Z000000F1,Z000000F2,
     1       Z000000F3,Z000000F4,Z000000F5,
     2       Z000000F6,Z000000F7,Z000000F8,Z000000F9/
      IF((K.GE.-999).AND.(K.LE.999)) GO TO 4
      LAB = STARS
      GO TO 5
    4 LAB=0
C
      IN=IABS(K)
      IT=10
      DO 1 I=1,3
      IA=I
      J=MOD(IN,IT) + 1
      LAB = LAB + N(J)* 256**(I-1)
C     WRITE(6,9) LAB
C   9 FORMAT(10X,I10)
      IN=IN/IT
      IF (IN.LE.0) GO TO 2
    1 CONTINUE
C
C
    2 IS=BL
      IF (K.LT.0) IS = M
      LAB = LAB + 256**(IA)*IS
```

```
C      WRITE(6,9) LAB
C
       IF (IA.GE.3) GO TO 5
       IB=IA+1
       DO 3 I=IB, 3
     3 LAB = LAB + 256**(I)*BL
     5 RETURN
       END
       REAL FUNCTION G(X)
       G=SIN(X)
       RETURN
       END
C
       REAL FUNCTION F(X)
       F=4.88/1.563*COS(X)
       RETURN
       END


       SUBROUTINE  DE( Z,ZDOT)
       REAL*8 Z(2),ZDOT(2)
       ZDOT(1)=Z(2)
       ZDOT(2)=-4.88/1.563*DCOS(Z(1))*Z(2)-DSIN(Z(1))
       RETURN
       END
INPUT DATA
         2.        3
'BOX H Y.L.HSIUNG                         JUNE 1969'
'PHASE PORTRAIT OF 2ND ORDER DIFF. EQUATION        '
       -20.         2.        20.
        -1.       0.25         1.
      -100.        40.       100.
      .0625         8
         0.        20.         0.         2.
         0.        20.         0.         4.
         0.        20.         0.         6.
         0.        20.         0.         8.
         0.        20.         0.        10.
         0.        20.         0.        12.
         0.        20.         0.        14.
         0.        20.         0.        16.
```

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13. ABSTRACT

The phase-plane method is a graphical method for linear and non-linear second-order systems. There are many techniques for obtaining the phase portrait which consists of a number of phase trajectories on the phase plane. From the summary and discussion the isocline method is a most general and useful method. The only difficulty is in the labor required. Solution by digital computer reduces this difficulty and makes the isocline method a much more useful method in non-linear systems analysis and design applications.

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